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THE FEE, A NEW TUNABLE HIGH RESOLUTION SPECTRUM ESTIMATOR.(U)
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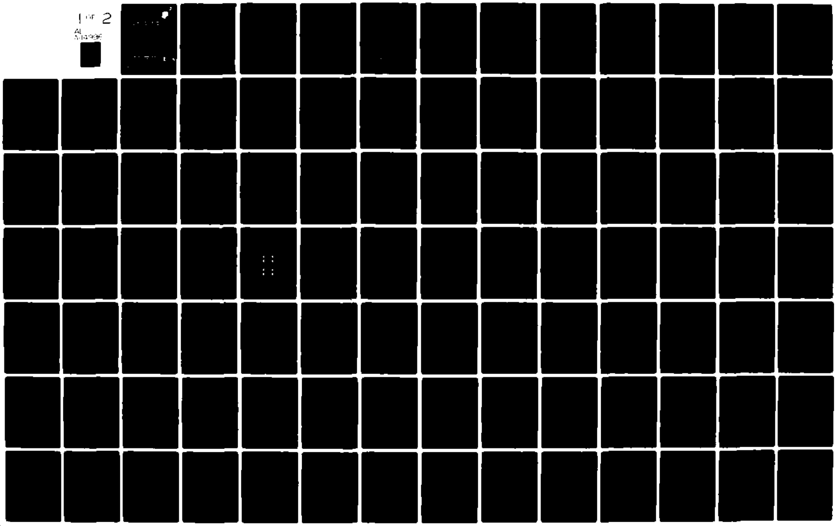

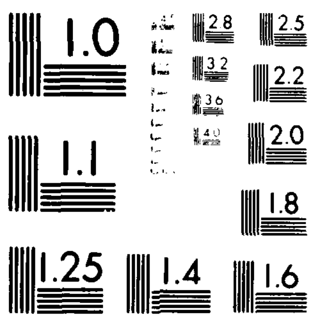

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February 1982

THE FEE, A NEW TUNABLE HIGH RESOLUTION SPECTRUM ESTIMATOR

Haywood E. Webb, Jr.

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ROME AIR DEVELOPMENT CENTER
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report consists of three parts:

1. A circuit theory model of the spectral estimation problem is given. The flat echo estimator is defined. Some of its properties are explored.

2. Part two addresses a software implementation in an interactive mode. Examples are given.

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3. Part three generalizes the scaler problem of Part 1 to the multi-dimensional vector cases.

PREFACE

In problems of the sort treated here, one is given a projection (segment) of a non-negative definite function. The problem is to extend (continue) the function over all its domain in such a way that it is even and non-negative definite. Subsequently, the Fourier transform of an even and non-negative definite function is even and non-negative, we have generated a spectrum. In the problem of spectral estimation, the given segment of data is from a covariance function, so that the Fourier transform of it together with its extension, is a power spectrum. All solutions considered here have the property of consistency, that is, the inverse Fourier transform of the estimated spectrum reproduces itself over the given data interval.

This report represents work accomplished under the technical direction of the undersigned with the support of the RADC Laboratory Director's fund.

HAYWOOD E. WEBB, Jr.

PART 1

BACKGROUND THEORY AND DERIVATION

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PART II

A DIGITAL IMPLEMENTATION

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PART III

MULTI CHANNEL CASE

by

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PART IV

SOME OBSERVATIONS AND
SUGGESTIONS FOR NEW WORK

by

HAYWOOD E. WEBB Jr.

PART I - BACKGROUND THEORY AND DERIVATION

by

D.C. YOULA

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ABSTRACT

The FEE, or Flat-Echo Estimator, represents an entirely new solution to the problem of spectrum recovery from a given finite set of error-free covariance samples. In concept it is based squarely on frequency-domain network-theoretic ideas and its principal features are the following.

1) Numerical robustness, as defined in this report, can be assigned in advance by the adjustment of a single parameter.

2) It is possible to "tune" the estimator to produce selective amplification of a finite number of desired frequencies without impairing either its interpolatory character or its numerical robustness. Enhanced resolution is to be expected.

3) For the parameter setting which yields maximum robustness, the FEE coincides with the MEE, the Maximum-Entropy Estimator.

In addition to providing a rather leisurely account of some of the network ideas behind the FEE, this research report also contains two appendices, A and B. The first supplies an explicit and compact formula for the class of all interpolatory spectral densities, while the second is devoted to the design of minimum-degree regular all-passes with prescribed phase at a given finite set of frequencies.

I. INTRODUCTION AND PRELIMINARY RESULTS

Suppose that we have a discrete-time random process x_t where "t" can traverse all positive and negative integers. Let us also suppose that the process is zero-mean and second-order stationary. Then,[1]

$$E(x_t\bar{x}_{t+k}) = C(k) \ , \quad |k| = 0 \rightarrow \infty \ , \tag{1}$$

is the associated covariance function. Clearly, for all integers k,

$$C(-k) = E(x_t\bar{x}_{t-k}) = E(x_{t+k}\bar{x}_t) = \bar{C}(k) \ . \tag{2}$$

As is well known [1],

$$C(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\theta} dF(\theta) \tag{3}$$

where $F(\theta)$ is monotone-nondecreasing, continuous from the left and of bounded-variation:

$$\mathrm{Var}\,F \equiv \int_{-\pi}^{\pi} dF(\theta) = 2\pi C(0) < \infty \ . \tag{4}$$

Further, the derivative

$$K(\theta) = \frac{dF(\theta)}{d\theta} \geq 0 \tag{5}$$

exists for almost all θ, is L_1 over $-\pi \leq \theta \leq \pi$ and defines the spectral density of the process. Evidently, if $F(\theta)$ is absolutely continuous,

$$C(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\theta} K(\theta) d\theta \ . \tag{6}$$

Let us assume first that $F(\theta)$ is absolutely continuous and that $K(\theta)$ satisfies the Paley-Wiener criterion,

$$\int_{-\pi}^{\pi} \ln K(\theta) d\theta > -\infty \ . \tag{7}$$

---

(1) $\bar{a}$ = complex conjugate of a.

Then, there exists [1] a unique function

$$B(z) = \sum_{r=0}^{\infty} b_r z^r \ , \quad b_0 > 0 \quad , \tag{8}$$

which is analytic and devoid of zeros in $|z| < 1$ and is such that[2]

$$\sum_{r=0}^{\infty} |b_r|^2 < \infty \tag{9}$$

and[3]

$$K(\theta) = |B(e^{j\theta})|^2 \ , \quad \text{a.e.} \tag{10}$$

Moreover, x_t can be realized as the output of a causal digital filter with a square-summable impulse response[4]

$$h_r = \overline{b}_r \ , \quad r = 0 \rightarrow \infty \quad , \tag{11}$$

driven by an appropriate white noise source sequence u_t:

$$x_t = \sum_{r=0}^{\infty} h_r u_{t-r} \ , \quad (-\infty < t < \infty) \ , \tag{12}$$

$$E(u_r) = 0, \ E(u_r \overline{u}_m) = \delta_{rm} \ , \quad (-\infty < r, \ m < \infty) \quad , \tag{13}$$

$$E(x_t \overline{x}_{t+k}) = C(k) = \sum_{r=0}^{\infty} \overline{h}_{r+k} h_r , \ (-\infty < k < \infty) \quad . \tag{14}$$

---

(2) $B(z)$ is of class H_2.

(3) $B(e^{j\theta}) \equiv \lim_{r \rightarrow 1-0} B(re^{j\theta})$ exists for almost all θ.

(4) $\sum_{r=0}^{\infty} |h_r|^2 = \sum_{r=0}^{\infty} |b_r|^2 < \infty$.

We can now pose and solve several problems in spectral estimation.

Problem 1. It is known that $F(\theta)$ is absolutely continuous and that the spectral density $K(\theta) = dF(\theta)/d\theta$ satisfies the Paley-Wiener condition (7). Under the assumption that the n+1 covariance samples $C(0), C(1), \ldots, C(n)$ are known absolutely and without error, determine, if it exists, a compatible $K(\theta)$ which maximizes h_o in the input-output model (12).

Solution. Let the numbers $C(k)$, $k = n+1 \to \infty$, constitute any admissible completion of the given data $C(0), C(1), \ldots, C(n)$ and let $K(\theta)$ be the corresponding spectral density.

Let

$$d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jk\theta} \ell n K(\theta) d\theta \quad , \tag{15}$$

$k = 0 \to \infty$, let $a_0 = 1$ and let the coefficients a_k, $k = 1 \to \infty$, be determined from the equation

$$\exp\left(\sum_{r=1}^{\infty} d_r z^r \right) = \sum_{k=0}^{\infty} a_k z^k \quad . \tag{16}$$

Then [1],

$$h_r = \bar{a}_r \exp(d_0/2) \ , \quad r = 0 \to \infty \quad . \tag{17}$$

Theorem 1. To construct an admissible completion $C(k)$, $k = n+1 \to \infty$, which maximizes

$$h_o = \exp(d_0/2) \quad , \tag{18}$$

is equivalent to choosing $K(\theta)$ to maximize

$$d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ell n K(\theta) d\theta \quad . \tag{19}$$

But [9] this leads immediately to the MEE, the unique maximum-entropy estimator, $K_{ME}(\theta)$, Eqs. (73) and (74).

Comment 1. What does it mean to maximize h_o and is it also possible to select spectral density estimators which maximize other meaningful

functionals defined on the impulse response sequence $\{h_r\}$? These questions deserve to be studied and one possible approach is to combine (17) with the following explicit formulas for the a_k's and d_k's [2]:

$$a_1 = d_1, \; a_2 = d_2 + \frac{d_1^2}{2}, \; a_3 = d_3 + d_2 d_1 + \frac{d_1^3}{6} \tag{20}$$

and in general,[5]

$$a_r = \frac{1}{r!} \det \begin{bmatrix} d_1 & -1 & 0 & \cdots & 0 \\ 2d_2 & d_1 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (r-1)d_{r-1} & (r-2)d_{r-2} & (r-3)d_{r-3} & \cdots d_1, & -(r-1) \\ rd_r & (r-1)d_{r-1} & (r-2)d_{r-2} & \cdots 2d_2, & d_1 \end{bmatrix}, \quad r \geq 1, \tag{21}$$

$$= d_r + d_1 d_{r-1} + \cdots, \quad r > 2 \quad . \tag{22}$$

Problem 2. Let $F(\theta)$ and $K(\theta)$ be subject to the same constraints as in problem 1 and assume once again that the covariance samples $C(0), C(1), \ldots, C(n)$ are known absolutely and without error. Provide, if feasible, a geometric time-domain interpretation of the process whereby every admissible completion $C(k)$, $k = n+1 \to \infty$, is generated.

Solution. As is well known [2], under the given hypotheses,[6]

$$\Delta_k = \begin{vmatrix} C(0) & C(1) & \cdots & C(k) \\ C(-1) & C(0) & \cdots & C(k-1) \\ \cdots & \cdots & \cdots & \cdots \\ C(-k) & C(-k+1) & \cdots & C(0) \end{vmatrix} > 0, \quad k = 0 \to \infty, \tag{23}$$

for any possible admissible extension $C(k)$, $k = n+1 \to \infty$, where

---

(5) $\det A \equiv |A|$ = determinant of square matrix A.

(6) The Paley-Wiener condition (7) precludes the possibility $\Delta_k = 0$.

$$C(-k) \equiv \bar{C}(k) \ . \tag{24}$$

Thus, at the first step, C(n+1) must be chosen from the set of values ξ which satisfy the inequality

$$\Delta_{n+1}(\xi) \equiv \begin{vmatrix} C(0) & C(1) & \ldots C(n) & \xi \\ C(-1) & C(0) & \ldots C(n-1) & C(n) \\ \cdots & \cdots & \cdots & \cdots \\ C(-n) & C(-n+1) & \ldots C(0) & C(1) \\ \bar{\xi} & C(-n) & \ldots C(-1) & C(0) \end{vmatrix} > 0 \ . \tag{25}$$

However, it is not difficult to show [2] that this set is the interior of a circle $\odot_n$ with radius

$$r_n = \frac{\Delta_n}{\Delta_{n-1}} \tag{26}$$

and center

$$\xi_n = \frac{(-1)^{n-1}}{\Delta_{n-1}} \cdot \begin{vmatrix} C(1) & C(2) & C(n) & 0 \\ C(0) & C(1) & \ldots C(n-1) & C(n) \\ \cdots & \cdots & \cdots & \cdots \\ C(-n+1) & C(-n+2) & \ldots C(0) & C(1) \end{vmatrix} \ . \tag{27}$$

In fact, if $\Delta_k{}'$ denotes the minor of Δ_k that is obtained by striking out its first column and last row, it follows from a fundamental determinantal result(7) applied to $\Delta_{n+1}(\xi)$ that

$$|\Delta_{n+1}{}'(\xi)|^2 = \Delta_n^2 - \Delta_{n-1}\Delta_{n+1}(\xi) \ . \tag{28}$$

But as is clear from (25) and (26),

$$\Delta_{n+1}{}'(\xi) = (-1)^n \Delta_{n-1} \cdot (\xi - \xi_n) \tag{29}$$

---

(7)If D is any determinant, and S is the second minor obtained from it by striking out its i^{th} and k^{th} rows and its j^{th} and ℓ^{th} columns, and if we denote by A_{ij} the cofactor of the element which stands in the i^{th} row and j^{th} column of D, then [3],

$$\begin{vmatrix} A_{ij} & A_{i\ell} \\ A_{kj} & A_{k\ell} \end{vmatrix} = (-1)^{i+j+k+\ell} DS \ . \tag{29a}$$

and (28) yields,

$$|\xi - \xi_n|^2 = r_n^2 - \frac{\Delta_{n+1}(\xi)}{\Delta_{n-1}} < r_n^2 \tag{30}$$

since $\Delta_{n+1}(\xi)$ and Δ_{n-1} are both positive.

Once having chosen $C(n+1) = \xi$ consistent with (30), $C(n+2)$ is then selected from the interior of a circle of radius $r_{n+1} = \Delta_{n+1}/\Delta_n$ and center ξ_{n+1} given by (27) with n replaced by n+1, *etcetera*. Evidently, except for r_n and ξ_n which are uniquely determined by the prescribed data C(0), C(1), ..., C(n), all successive r_k and ξ_k depend on the particular rule used to pick C(k+1) from the interior of Θ_k, $k = n+1 \to \infty$.

Theorem 2. Under the conditions prevailing in problem 1, the following is true.

1) Let C(k), $k = n+1 \to \infty$, denote any admissible completion of the initial covariance data C(0), C(1), ..., C(n) and let $K(\theta)$ be the corresponding spectral density estimator. Let $K(\theta) = |B(e^{j\theta})|^2$ where B(z), Eq. (8), is the associate Wiener-Hopf factor. Then, the sequence of radii r_k, $k = n \to \infty$, generated by this completion is always monotone-nonincreasing and

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \frac{\Delta_k}{\Delta_{k-1}} = b_o^2 = h_o^2 > 0 \quad . \tag{31}$$

In addition, for the sequence of centers, ξ_k, $k = n \to \infty$,

$$\limsup_{k\to\infty} |\xi_k| \leq b_o^2 \leq \frac{\Delta_n}{\Delta_{n-1}} \quad . \tag{32}$$

2) The maximum-entropy estimator $K_{ME}(\theta)$ is the (unique) spectral density that is obtained by identifying each C(k+1) in the completion with the center of circle Θ_k:

$$C(k+1) = \xi_k \quad , \quad k = n \to \infty \quad . \tag{33}$$

Under this ME rule,

$$\lim_{k\to\infty} \xi_k = 0 \tag{34}$$

and all circles Θ_k possess the <u>same maximum possible</u> radius r_n:

$$r_n = \frac{\Delta_n}{\Delta_{n-1}} = \frac{\Delta_k}{\Delta_{k-1}} = r_k \quad , \tag{35}$$

$$k = n \to \infty \quad .$$

Proof (in outline). Given any admissible completion and its corresponding spectral density estimator $K(\theta) = |B(e^{j\theta})|^2$ where $B(z)$ is the associate Wiener-Hopf factor, it is well known [4] that

$$\lim_{k\to\infty} \frac{\Delta_k}{\Delta_{k-1}} = b_o^2 > 0 \quad . \tag{36}$$

Furthermore, from (28) with n replaced by k,

$$r_{k+1} = \frac{\Delta_{k+1}}{\Delta_k} = \frac{\Delta_k}{\Delta_{k-1}} \left(1 - \left|\frac{\Delta_{k+1'}}{\Delta_k}\right|^2\right) \leq \frac{\Delta_k}{\Delta_{k-1}} = r_k \quad , \tag{37}$$

and it is clear that the sequence r_k, for $k = n \to \infty$, is monotone-nonincreasing. Thus,

$$b_o^2 = \lim_{k\to\infty} \frac{\Delta_k}{\Delta_{k-1}} \leq \frac{\Delta_n}{\Delta_{n-1}} = r_n \tag{38}$$

with equality iff

$$\Delta_{k+1'} = 0, \quad k = n \to \infty \quad . \tag{39}$$

Since $K(\theta)$ is integrable,

$$\lim_{k\to\infty} C(k+1) = 0 \tag{40}$$

which implies that all circles Θ_k contain the origin as an interior point for k sufficiently large. Hence, for large enough k,

$$|\xi_k| < r_k \leq r_n \tag{41}$$

so that

$$\limsup_{k\to\infty} |\xi_k| \leq \lim_{k\to\infty} r_k = b_o^2 \leq r_n = \frac{\Delta_n}{\Delta_{n-1}} \quad . \tag{42}$$

To establish 2), observe that (39) and (29) lead to the conclusion that b_0 assumes its maximum possible value

$$(b_o)_{MAX} = \sqrt{\frac{\Delta_n}{\Delta_{n-1}}} \tag{43}$$

iff for every $k = n \to \infty$,

$$\xi = C(k+1) = \xi_k \quad , \tag{44}$$

the center of circle Θ_k. But according to (11) and (18), $b_0 = \exp(d_0/2)$ so that $(b_0)_{MAX}$ is necessarily paired with the estimator $K(\theta) = K_{ME}(\theta)$ which maximizes the entropy

$$d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln K(\theta) d\theta \quad . \tag{45}$$

Lastly, since the ME completion rule identifies ξ_k with $C(k+1)$ for every $k \geq n$, it is evident that

$$\lim_{k\to\infty} \xi_k = \lim_{k\to\infty} C(k+1) = 0 \quad , \tag{46}$$

Q.E.D.

Comment 2. Theorem 2 suggests that the ME estimator $K_{ME}(\theta)$ achieves good performance because its completion rule possesses a kind of optimal robustness against loss of positive-definiteness. Or, stated somewhat differently, since each additional covariance sample is chosen at the center of its circle of admissibility, it appears plausible to conjecture, on the grounds of symmetry, that the ME rule possesses a fair degree of self-correction in the face of (unsystematic) sample error. Admittedly, these considerations are heuristic in nature but they seem to be reinforced even more by the realization that the initial data $C(0), C(1), \ldots, C(n)$ must actually be estimated from samples of the random process x_t itself.

Problem 3. Under the assumption that $C(k)$ is real, place the general problem of spectral estimation from a prescribed set of error-free covariance samples $C(0), C(1), \ldots, C(n)$, in a network-theoretic setting involving positive-real functions.

Solution. First, it is easily seen that C(k) real implies that the function

$$Z(z) = C(0) + \sum_{k=1}^{\infty} 2C(k)z^k \tag{47}$$

is positive-real (pr) in $|z| < 1$, i.e., Z(z) is real for z real and

$$\operatorname{Re} Z(z) \equiv \frac{Z(z) + \bar{Z}(z)}{2} \geq 0 \; , \quad |z| < 1 \quad . \tag{48}$$

The proof is quite simple.

In fact, it follows readily from the reality of C(k) and (3) that[8]

$$Z(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\theta}+z}{e^{j\theta}-z} \, dF(\theta) \; , \quad |z| < 1 \quad . \tag{49}$$

Hence,

$$\operatorname{Re} Z(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{j\theta}-z|^2} \, dF(\theta) \geq 0, \quad |z| < 1 \; , \tag{50}$$

since $dF(\theta) \geq 0$, Q.E.D.

It can be shown [5], that for almost all θ, the "interior" radial limit

$$Z(e^{j\theta}) \equiv \lim_{r \to 1-0} Z(re^{j\theta}) \tag{51}$$

exists and that

$$K(\theta) = \operatorname{Re} Z(e^{j\theta}) \quad . \tag{52}$$

However, even more is true. Let $F_s(\theta)$ denote the singular part[9] of $F(\theta)$ and let the countable set of discontinuities of $F(\theta)$ occur at $\theta = \theta_i$ with jumps $\rho_i > 0$, $i = 1 \to \infty$. Then, (49) may be rewritten as

$$Z(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\theta}+z}{e^{j\theta}-z} \, dF_s(\theta) + \frac{1}{2\pi} \sum_{i=1}^{\infty} \rho_i \frac{e^{j\theta_i}+z}{e^{j\theta_i}-z} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\theta}+z}{e^{j\theta}-z} K(\theta) d\theta \tag{53}$$

$$= \qquad Z_s(z) \qquad + \qquad Z_F(z) \qquad + \qquad Z_a(z) \qquad . \tag{54}$$

---

(8) The interchange of summation and integration is justified by dominated convergence.

(9) $F_s(\theta)$ is monotone-nondecreasing and its derivative equals zero almost everywhere [6].

Clearly,

$$\frac{\rho_i}{\pi} = \underset{z \to e^{j\theta_i}}{\text{limit}} \, (1 - ze^{-j\theta_i})\, Z(z) \; , \quad i = 1 \to \infty \quad , \tag{55}$$

and the ρ_i's emerge as residues of $Z(z)$ at its poles $z_i = e^{j\theta_i}$ (all simple) on $|z| = 1$. These poles are revealed by the presence of the Foster reactance

$$Z_F(z) = \frac{1}{2\pi} \sum_{i=1}^{\infty} \rho_i \frac{e^{j\theta_i} + z}{e^{j\theta_i} - z} \quad . \tag{56}$$

Of course,

$$\operatorname{Re} Z_F(e^{j\theta}) \equiv 0 \tag{57}$$

and since $F(\theta)$ is of bounded-variation,

$$\sum_{i=1}^{\infty} \rho_i < \infty \tag{58}$$

so that the infinite sum in (56) is well-defined.

Thus, given $C(k)$, $k = 0 \to n$, the problem of spectral estimation is equivalent to that of generating all pr functions[10]

$$Z(z) = C(0) + 2C(1)z + \ldots + 2C(n)z^n + O(z^{n+1}) \quad . \tag{59}$$

We shall now present an elementary solution based on the synthesis of ideal TEM transmission-line cascades [7, 8].

The Interpolatory Cascade. With the given data of real numbers $C(k)$, $k = 0 \to n$, construct the real symmetric $(n+1) \times (n+1)$ matrix

$$T_n = \begin{bmatrix} C(0) & C(1) & \ldots & C(n) \\ C(1) & C(0) & \ldots & C(n-1) \\ \cdots & \cdots & \cdots & \cdots \\ C(n) & C(n-1) & \ldots & C(0) \end{bmatrix} \quad . \tag{60}$$

[10] Phrased differently, find all pr function $Z(z)$ such that $d^k Z(0)/dz^k$, $k = 0 \to n$, are prescribed real numbers.

An admissible completion C(k), $k = n+1 \rightarrow \infty$, exists, if and only if $T_n \geq 0_{n+1}$.[11] Two cases must be distinguished when this criterion is satisfied.

Case 1: $\Delta_n = \det T_n > 0$. Then, rank $T_n = n+1$ and all interpolatory pr solutions Z(z) are constructed in the following manner.

Step 1. Let T_k denote the (k+1) x (k+1) Toeplitz matrix obtained by substituting k for n in (60), let

$$\Delta_k = \det T_k \, , \quad k \geq 0 \quad , \tag{61}$$

and let Δ_k' denote the minor of Δ_k obtained by striking out its first column and last row. Let N denote the 2-port reactance-cascade (Fig. 1) composed of n+1 ideal TEM lines all of the same 1-way delay $\tau > 0$ and positive characteristic impedances,

$$R_0 = C(0) \tag{62}$$

and

$$R_k = R_{k-1} \cdot \frac{1 + (-1)^{k-1} \cdot \dfrac{\Delta_k'}{\Delta_{k-1}}}{1 - (-1)^{k-1} \cdot \dfrac{\Delta_k'}{\Delta_{k-1}}} \, , \quad k = 1 \rightarrow n \; . \tag{63}$$

Then, under the identification[12]

$$z = e^{-2p\tau} \quad , \tag{64}$$

every admissible interpolatory solution Z(z) is obtained as the impedance seen looking into the input port of N with its output port closed on an arbitrary pr function W(z). (In Fig. 1, terminals cd are connected to ab.)

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(11) 1_k and 0_k denote the kxk identity and zero matrices, respectively. For any matrix A, $\bar{A}$, A' and $A^*(\equiv \bar{A}')$ denote its complex conjugate, transpose and adjoint. If $A(=A^*)$ is kxk and hermitean, $A > 0_k (\geq 0_k)$ means that it is positive-definite (nonnegative-definite). As usual $\underset{\sim}{x}$ = column-vector.

(12) $p = \sigma + j\omega$ is the standard complex-frequency variable used in network theory.

The quantities

$$s_k = \frac{R_k - R_{k-1}}{R_k + R_{k-1}} = (-1)^{k-1} \cdot \frac{\Delta_k'}{\Delta_{k-1}} \ , \quad k = 1 \to n \ , \qquad (65)$$

are the n junction reflection coefficients. Their physical significance is not difficult to grasp. Indeed, let the terminated structure depicted in Fig. 1 be excited by an input current source $i(t) = \delta(t)$. This impulse immediately launches a voltage impulse $C(0)\delta(t)$ of strength $a_{0t} = C(0)$ into line 0 which then travels towards line 1.

After τ secs., a_{0t} reaches the first junction separating line 0 and line 1 and part of it is reflected and part of it is transmitted.(13) If a_{1r} and a_{1t} denote the strengths of these reflected and transmitted impulses, respectively, conventional transmission line theory yields,

$$\left(\frac{a_{1r}}{a_{0t}}\right)^2 = s_1^2 \qquad (66)$$

and

$$\left(\frac{a_{1t}}{a_{0t}}\right)^2 = 1 - s_1^2 \ . \qquad (67)$$

In the same way, after τ more secs., a_{1t} reaches the second junction and is partly reflected and partly transmitted to generate a_{2r} and a_{2t} where

$$\left(\frac{a_{2r}}{a_{1t}}\right)^2 = s_2^2 \qquad (68)$$

and

$$\left(\frac{a_{2t}}{a_{1t}}\right)^2 = 1 - s_2^2 \ . \qquad (69)$$

In general therefore, upon reaching line k, $a_{(k-1)t}$ is divided into an impulse of strength a_{kr} reflected back into line k-1 and one of strength a_{kt} transmitted into line k. Clearly,

$$\left(\frac{a_{kr}}{a_{(k-1)t}}\right)^2 = s_k^2 \ , \quad \left(\frac{a_{kt}}{a_{(k-1)t}}\right)^2 = 1 - s_k^2 \ , \quad k=1 \to n \ . \qquad (70)$$

---

(13) It is convenient to label an impulse by its strength.

Moreover, since interaction with the passive load $W(z)$ commences only after the lapse of $(n+1)\tau$ secs., the input voltage response $v(t)$ over the time interval $0 \leq t < 2(n+1)\tau$ is determined solely by the $(n+1)$-line cascade. We can now relate the radii r_k to the squared amplitudes a_{kt}^2.

For according to Eqs. (26), (28) and (65),

$$1 - s_k^2 = 1 - \left(\frac{\Delta_k'}{\Delta_{k-1}}\right)^2 = \frac{\Delta_{k-2}\Delta_k}{\Delta_{k-1}^2} = \frac{r_k}{r_{k-1}}\ , \quad k=1\to n\ , \tag{71}$$

$$\Delta_{-1} \equiv 1\ .$$

Thus, since $r_0 = C(0) = a_{0t}$, a comparison of (71) with (70) immediately gives the important physical connection formula,

$$r_k = \frac{a_{kt}^2}{C(0)}\ , \quad k = 0 \to n\ . \tag{72}$$

Obviously then, since successive transmitted impulses can never increase in strength, the corresponding sequence of successive radii of the circles of admissibility must be monotone-nonincreasing.

It follows from this interpretation, that to avoid any decrease in circle radius for $k>n$, it is necessary that the last transmitted impulse in line n suffer no reflection upon reaching the load $W(z)$. But this is only possible if $W(z) = R_n$, the characteristic impedance of the last line in the $(n+1)$-line cascade.

Consequently, $K_{ME}(\theta)$ is the real part of the driving-point impedance of the $n+1$-line structure N closed on $W(z) = R_n$. Explicitly [9, 10, Appendix A],

$$K_{ME}(\theta) = \frac{1}{|P_n(e^{j\theta})|^2}\ , \tag{73}$$

where

$$P_n(z) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} C(0) & C(1) & \dots & C(n) \\ C(1) & C(0) & \dots & C(n-1) \\ \dots & \dots & \dots & \dots \\ C(n-1) & C(n-2) & \dots & C(1) \\ z^n & z^{n-1} & \dots & 1 \end{vmatrix} \tag{74}$$

is a degree n <u>strict-Hurwitz</u> polynomial.[14]

Case 2: $\Delta_n = \det T_n = 0$. Now rank $T_n < n+1$ and there exists a $\mu < n$ such that $\Delta_\mu = \det T_\mu > 0$ but $\Delta_{\mu+1} = \det T_{\mu+1} = 0$. The interpolatory pr function $Z(z)$ is <u>unique</u>, is <u>Foster</u> and is of degree $\mu+1 = \text{rank}\, T_n$. Thus, according to (53) and (54), $Z_s(z) = Z_a(z) \equiv 0$ and

$$Z(z) = Z_F(z) = \frac{1}{2\pi} \sum_{i=1}^{\mu+1} \rho_i \frac{e^{j\theta_i} + z}{e^{j\theta_i} - z} \quad . \tag{75}$$

The poles $z_i = e^{j\theta_i}$, and the residues ρ_i, $i = 1 \to \mu+1$, can be found in three steps:[15]

1) Find a real nontrivial column-vector solution $\underset{\sim}{x} = (x_0, x_1, \ldots, x_{\mu+1})'$ of the homogeneous equation

$$T_{\mu+1} \underset{\sim}{x} = \underset{\sim}{0} \quad . \tag{76}$$

(There exists only one such linearly independent $\underset{\sim}{x}$.)

2) The $\mu+1$ zeros of the polynomial

$$x(z) = x_0 + x_1 z + \cdots + x_{\mu+1} z^{\mu+1} \tag{77}$$

are all of unit modulus and are precisely the desired poles $z_i = e^{j\theta_i}$, $i = 1 \to \mu+1$.

3) With the $e^{j\theta_i}$'s known from 2), the quantities $\rho_1, \rho_2, \ldots, \rho_{\mu+1}$ are determined as the unique solutions of the linear system of the <u>Vandermonde</u> type,

$$C(k) = \sum_{i=1}^{\mu+1} \rho_i e^{-jk\theta_i} \quad , \quad k = 0 \to \mu \quad . \tag{77a}$$

A much more enlightening physical interpretation of case 2 is available. Namely, since $\Delta_\mu > 0$ implies that $T_\mu > 0_{\mu+1}$, then

$$\Delta_k > 0 \; , \quad k = 0 \to \mu \; . \tag{78}$$

---

(14) $P_n(z) \neq 0$, $|z| \leq 1$. Incidentally, observe that $P_n(z)$ is the determinant of the matrix that results by replacing the last row in T_n by $z^n, z^{n-1}, \ldots, z, 1$.

(15) This three-step rule goes back to F. Riesz [2] but its use in the context of spectral estimation is of recent vintage [11].

Hence, from (71),

$$s_k^2 < 1 \quad , \quad k = 1 \to \mu \; . \tag{79}$$

However, since $\Delta_{\mu+1} = 0$, $s_{\mu+1}^2 = 1$; i.e., $s_{\mu+1} = \pm 1$ and consequently, in view of (65), $R_{\mu+1} = \infty$ or 0. In other words, the interpolatory Z(z) is the input impedance of a cascade of $\mu+1$ TEM lines of characteristic impedances $R_0, R_1, \cdots, R_\mu$ closed on <u>either</u> an open or short circuit (but not both!). This explains why the process terminates and why Z(z) is unique. Alternatively, we can say that the choice of covariance sample $C(\mu+1)$ has been made from the <u>boundary</u> of the circle of admissibility Θ_μ and as a result the transient voltage impulse transmitted into line $\mu+1$ must be zero. This situation is realized only because the load that terminates line μ is either an open or a short. (We conclude that the two associated <u>real</u> sample values $C(\mu+1)$ are the two intersections of Θ_μ with the real axis.)[16]

In the next section we shall undertake a much deeper study of the properties of the spectral estimator, especially with regard to its dependence on the choice of pr load W(z). Nevertheless, for now we can assert that any selection of covariance sample made from the boundary of its circle of admissibility immediately forces Z(z) to be a unique Foster function of finite degree and C(k) to be a finite sum of <u>pure</u> tones ($F_s(\theta) = K(\theta) \equiv 0$):

$$C(k) = \frac{1}{2\pi} \sum_{i=1}^{\mu+1} \rho_i e^{-jk\theta_i} \quad . \tag{80}$$

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[16] Since the data C(k), $k = 0 \to \mu$, are real, the center of Θ_μ lies on the real axis.

II. "TUNED" SPECTRAL ESTIMATORS

From this point on it is assumed that the given data $C(k)$, $k = 0 \rightarrow n$, are real and error-free, that T_n is positive-definite and that $F(\theta)$ is absolutely-continuous. Thus, the objective is to estimate the spectral density $K(\theta) = dF(\theta)/d\theta$ and we restrict our attention exclusively to estimators defined by rational pr functions $W(z)$.

Let

$$\rho(z) = \frac{W(z) - R_n}{W(z) + R_n} \tag{81}$$

denote the reflection coefficient of the load normalized to R_n, the characteristic impedance of the last line. Since $W(z)$ is pr, $\rho(z)$ is bounded-real (br); i.e.,

$$|\rho(z)| \leq 1 \quad , \quad |z| < 1 \quad . \tag{82}$$

Let

$$Z_{ME}(z) = \frac{2Q_n(z)}{P_n(z)} \tag{83}$$

denote the input impedance $Z(z)$ with $W(z) = R_n$. Clearly, this corresponds to $\rho(z) \equiv 0$ and $P_n(z)$ is given by (74). It can be shown (Appendix A) that $Q_n(z)$ and $P_n(z)$ are two degree-n polynomials that satisfy the even-part condition,[17]

$$Q_n(z)P_{n*}(z) + Q_{n*}(z)P_n(z) = 1 \quad . \tag{84}$$

It is also a consequence of the same analysis that the even part of the impedance $Z(z)$ seen looking into the n+1-line cascade with the output closed on a general pr termination $W(z)$ is given by

$$\frac{Z(z) + Z_*(z)}{2} = \frac{1 - \rho(z)\rho_*(z)}{D_n(z)D_{n*}(z)} \tag{85}$$

where

$$D_n(z) = P_n(z) - z\rho(z)\tilde{P}_n(z) \tag{86}$$

(17) $A_*(z) \equiv A(1/z)$ for $A(z)$ real and rational.

and[18]

$$\tilde{P}_n(z) \equiv z^n P_{n*}(z) \quad . \tag{87}$$

Consequently, $Z(z)$ generates the spectral estimator

$$K(\theta) = \frac{1 - |\rho(e^{j\theta})|^2}{|D_n(e^{j\theta})|^2} \quad . \tag{88}$$

Observe that if $\rho(z) \equiv 0$, we recover $K_{ME}(\theta)$, Eq. (73). Clearly, under the assumption that $F(\theta)$ is absolutely continuous, $|\rho(e^{j\theta})| \not\equiv 1$ and $W(z)$ is non-Foster. In general, an estimator is said to be tuned if the associated reflection coefficient $\rho(z)$ is nonconstant.

Let

$$1 - \rho(z)\rho_*(z) = \Gamma(z)\Gamma_*(z) \tag{89}$$

where $\Gamma(z)$ is a real rational function analytic together with its inverse in $|z| < 1$. (This Wiener-Hopf factor is uniquely determined by the normalization condition $\Gamma(0) > 0$. Furthermore, $\Gamma(0) < 1$ unless $\rho(z) \equiv 0$.) Hence, from (85),

$$K(\theta) = |B(e^{j\theta})|^2 \tag{90}$$

where

$$B(z) = \frac{\Gamma(z)}{D_n(z)} \quad . \tag{91}$$

Since $D_n(z)$ is free of zeros in $|z| < 1$,[19] $B(z)$ is the Wiener factor for $K(\theta)$!

Theorem 3. Let the pr function $W(z)$ generate the spectral estimator $K(\theta)$, given the error-free covariance data $C(k)$, $k = 0 \rightarrow n$. Let $\{r_k\}$ denote the sequence of radii of the corresponding circles of admissibility. As we

---

(18) $\tilde{P}_n(z)$ is the polynomial reciprocal to $P_n(z)$.

(19) $P_n(z)$ is strict Hurwitz and

$$1 - z\rho(z)\frac{\tilde{P}_n(z)}{P_n(z)}$$

is positive-real.

know, $r_k \downarrow r_\infty = b_0^2$ where $b_0 = B(0)$. In addition, from (11) and (17),

$$d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln K(\theta) d\theta = 2 \ln b_0 \quad . \tag{92}$$

Thus, (91) yields[20]

$$1) \quad r_\infty(K) = \Gamma^2(0) \cdot \frac{\Delta_n}{\Delta_{n-1}} = \Gamma^2(0) \cdot r_\infty(K_{ME}) \leq r_\infty(K_{ME}) \tag{93}$$

with equality iff $K(\theta) = K_{ME}(\theta)$.

$$2) \quad \text{Entropy}(K) = \text{entropy}(K_{ME}) - \ln \frac{1}{\Gamma^2(0)} \leq \text{Entropy}(K_{ME}) \tag{94}$$

with equality iff $K(\theta) = K_{ME}(\theta)$. Or, as an alternative to (94) we may also write,

$$\text{Entropy}(K) = \text{entropy}(K_{ME}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \frac{1}{1 - |\rho(e^{j\theta})|^2} d\theta \quad . \tag{95}$$

We are now in a position to describe an entirely new type of spectral estimator with the capability of controlling both resolution and robustness (as measured by r_∞) in a parametric fashion. To this end, observe that Eq. (95) permits us to define

$$\frac{1}{2\pi} \cdot \ln \frac{1}{1 - |\rho(e^{j\theta})|^2} \tag{96}$$

to be the entropy echo-loss density at angle θ.[21]

A flat-echo estimator (an FEE) is one for which the entropy echo-loss density is independent of θ. According to (96), this requires that

$$|\rho(e^{j\theta})| = \text{constant} \; . \tag{97}$$

But this is possible iff

$$\rho(z) = \mu d(z) \tag{98}$$

---

(20) Bear in mind that $D_n(0) = P_n(0)$.

(21) The phrase "echo-loss" is taken from filter theory [7].

where μ is a constant constrained to lie in the range

$$0 \leq \mu < 1 \quad , \tag{99}$$

and $d(z)$ is a regular all-pass; i.e., $d(z)$ is a real-rational function analytic in $|z| \leq 1$ which satisfies the paraunitary condition

$$d(z)d_{*}(z) = 1 \quad . \tag{100}$$

III. THE DESIGN OF TUNED FEE'S

Let us consider the properties of an FEE defined by the reflection coefficient $\rho(z) = \mu d(z)$, $d(z)$ a regular all-pass and $0 \leq \mu < 1$. Then,

$$\Gamma(z) = \sqrt{1 - \mu^2} \quad , \tag{101}$$

$$r_\infty(K_{FE}) = (1 - \mu^2) \cdot r_\infty(K_{ME}) \tag{102}$$

and $K_{FE}(\theta) = |B(e^{j\theta})|^2$ where

$$B(z) = \frac{\sqrt{1-\mu^2}}{P_n(z) - \mu z d(z) \tilde{P}_n(z)} \tag{103}$$

is the associate Wiener-Hopf factor. Thus,

$$K_{FE}(\theta) = K_{ME}(\theta) \cdot \frac{1-\mu^2}{|1-\mu e^{j\theta} d(e^{j\theta}) U_n(e^{j\theta})|^2} \tag{104}$$

where

$$U_n(z) = \frac{\tilde{P}_n(z)}{P_n(z)} \tag{105}$$

is <u>also</u> a regular all-pass (because $P_n(z)$ is strict-Hurwitz). Clearly, $U_n(z)$ is uniquely determined by the prescribed data $C(k)$, $k = 0 \rightarrow n$.

We can now list some of the anticipated advantages of an FEE.

A_1) Numerical robustness, as measured by r_∞, is <u>assignable in advance</u> by fixing the constant μ and is <u>unaffected</u> by <u>any</u> subsequent tuning.

A_2) On $|z| = 1$, $d(z)$ and $U_n(z)$ are both of unit modulus so that

$$d(e^{j\theta}) = \exp[j\psi(\theta)] \tag{106}$$

and

$$U_n(e^{j\theta}) = \exp[j\beta_n(\theta)] \quad , \tag{107}$$

where $\psi(\theta)$ and $\beta_n(\theta)$ are real functions of θ. Thus, if

$$\varphi(\theta) = \psi(\theta) + \beta_n(\theta) + \theta \quad , \tag{108}$$

the factor that multiplies $K_{ME}(\theta)$ in Eq. (104) reduces to

$$\frac{1-\mu^2}{1-2\mu\cos\varphi(\theta)+\mu^2} \quad . \tag{109}$$

Consequently,

$$\frac{1-\mu}{1+\mu} \leq \frac{K_{FE}(\theta)}{K_{ME}(\theta)} \leq \frac{1+\mu}{1-\mu} \tag{110}$$

and any such ratio can be realized at one or several prescribed values of θ by an appropriate choice of all-pass d(z). The FEE is therefore capable of providing selective controlled magnification of different parts of the spectrum. In our opinion, if used properly, this estimator could prove enormously valuable in situations requiring exceptional resolution. (The design of d(z) is one of phase equalization with a regular all-pass and is explained in detail in Appendix B.)

A_3) The poles of d(z) are zeros of B(z). Hence, the FEE not only includes the MEE as a limiting case ($\mu = 0$ = maximum robustness), but also yields exact model matching for a broader class of random processes than those generated by pure finite-order autoregressive schemes.[22]

A_4) $K_{FE}(\theta) > 0$, $-\pi \leq \theta \leq \pi$, a property that makes good physical sense whenever an exact spectral null is highly unlikely. (Such is usually the case if x_t is the sum of two or more independent random processes.)

Appendix B contains design formulas for a singly and doubly-tuned FEE. These realize maximum gain $(1+\mu)/(1-\mu)$, at one and two prescribed values of θ, respectively.

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[22] The only zero of

$$B_{ME}(z) = \frac{1}{P_n(z)}$$

is $z = \infty$.

APPENDIX A

NETWORK FOUNDATIONS

Let the reactance 2-port N′ shown in Fig. 1 be described by its 2x2 scattering matrix S(p) normalized to R_n on the right and R_0 on the left. Then [7],

$$S(p) = \frac{1}{g(z)}\left[\begin{array}{c|c} h(z) & e^{-np\tau} \\ \hline e^{-np\tau} & -z^n h_*(z) \end{array}\right] \equiv \left[\begin{array}{c|c} s_{11}(p) & s_{12}(p) \\ \hline s_{12}(p) & s_{22}(p) \end{array}\right] \tag{A1}$$

where $z = e^{-2p\tau}$ and h(z), g(z) are two real polynomials of degree n in z, the latter being strict-Hurwitz. Moreover,

$$g(z)g_*(z) - h(z)h_*(z) = 1 \tag{A2}$$

Let a load with pr driving-point impedance W(z) close terminals ab. Then, the load seen from terminals ef has a reflection coefficient normalized to R_n given by

$$s_\ell(z) = z\rho(z) = z \cdot \frac{W(z) - R_n}{W(z) + R_n} \quad . \tag{A3}$$

Hence [12],

$$s = s_{11} + \frac{s_{12}s_{21}s_\ell}{1 - s_{22}s_\ell} \tag{A4}$$

is the resultant input reflection coefficient normalized to R_0. Making use of (A1) and (A2) we obtain,

$$s(z) = \frac{h(z) + z\rho(z)\tilde{g}(z)}{g(z) + z\rho(z)\tilde{h}(z)} \tag{A5}$$

where $\tilde{g} = z^n g_*$ and $\tilde{h} = z^n h_*$.

Consequently, the input impedance is given by

$$Z(z) = R_o \cdot \frac{1+s(z)}{1-s(z)} = \frac{2(Q_n + z\rho\tilde{Q}_n)}{P_n - z\rho\tilde{P}_n} \tag{A6}$$

where

$$\frac{Q_n(z)}{\sqrt{R_0}} = \frac{g(z)+h(z)}{2} \quad , \tag{A7}$$

$$P_n(z) = \frac{g(z)-h(z)}{\sqrt{R_0}} \quad . \tag{A8}$$

Note, that with $W(z) = R_n$, $\rho(z) \equiv 0$ and

$$Z(z) = Z_{ME}(z) = \frac{2Q_n(z)}{P_n(z)} \quad . \tag{A9}$$

Lastly, by a direct calculation,

$$Q_n P_{n*} + Q_{n*} P_n = 1 \tag{A10}$$

so that (A6) yields (details omitted),

$$\frac{Z+Z_*}{2} = \frac{1-\rho\rho_*}{(P_n - z\rho\tilde{P}_n)(P_n - z\rho\tilde{P}_n)_*} \quad . \tag{A11}$$

Thus,

$$K(\theta) = \frac{Z(e^{j\theta}) + Z_*(e^{j\theta})}{2} = \frac{1-|\rho(e^{j\theta})|^2}{|D_n(e^{j\theta})|^2} \tag{A12}$$

in which

$$D_n(z) = P_n(z) - z\rho(z)\tilde{P}_n(z) \quad . \tag{A13}$$

Naturally, if $\rho(z) \equiv 0$,

$$K(\theta) = K_{ME}(\theta) = \frac{1}{|P_n(e^{j\theta})|^2} \tag{A14}$$

and for the sake of completeness we shall verify the determinantal expression (74) for $P_n(z)$.

From (A9) and (A10),

$$\frac{Z_{ME}(z) + Z_{ME*}(z)}{2} = \frac{Q_n(z)}{P_n(z)} + \frac{Q_{n*}(z)}{P_{n*}(z)} = \frac{1}{P_n(z)P_{n*}(z)} \quad . \tag{A15}$$

Let

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \quad a_0 > 0 \ . \tag{A16}$$

Then, since

$$Z_{ME}(z) = C(0) + \sum_{k=1}^{n} 2C(k)z^k + O(z^{n+1}) \ , \tag{A17}$$

(A15) and (A16) yield

$$\left(C(0) + \sum_{k=1}^{\infty} C(k)z^{-k} + \sum_{k=1}^{\infty} C(k)z^k\right)\left(a_n + a_{n-1}z + \cdots + a_0 z^n\right) = \frac{z^n}{P_n(z)} \ . \tag{A18}$$

It follows easily from (A10) that $P_n(z) \neq 0$, $|z| \leq 1$, so that $1/P_n(z)$ admits a power series expansion

$$b_0 + b_1 z + b_2 z^2 + \cdots \tag{A19}$$

whose radius of convergence is greater than unity. Of course,

$$b_0 = \frac{1}{P_n(0)} = \frac{1}{a_0} > 0 \quad . \tag{A20}$$

Consequently, we may compare the coefficients of $1, z, z^2, \ldots, z^n$ on both sides of (A18) to obtain

$$T_n \underset{\sim}{a} = b_o \underset{\sim}{e}_n \tag{A21}$$

where the Toeplitz matrix T_n is given by (60),

$$\underset{\sim}{a} = (a_n, a_{n-1}, \ldots, a_0)' \tag{A22}$$

and

$$\underset{\sim}{e}_n = (0, 0, \ldots, 1)' \tag{A23}$$

is composed of n 0's and a single 1 in the last row. Thus,

$$P_n(z) = (z^n, z^{n-1}, \ldots, 1)\,\underset{\sim}{a} = b_0(z^n, z^{n-1}, \ldots, 1)T_n^{-1}\,\underset{\sim}{e}_n \qquad \text{(A24)}$$

$$= \frac{1}{a_0\Delta_n}\begin{vmatrix} C(0) & C(1) & \ldots\; C(n) \\ C(1) & C(0) & \ldots\; C(n-1) \\ \cdots & \cdots & \cdots \\ C(n-1) & C(n-2) & \ldots\; C(1) \\ z^n & z^{n-1} & \ldots\; 1 \end{vmatrix} . \qquad \text{(A25)}$$

However, if we now use Cramer's rule to solve Eq. (A21) for a_0 we obtain

$$a_0 = \sqrt{\frac{\Delta_{n-1}}{\Delta_n}} \qquad \text{(A26)}$$

and (A25) immediately reduces to (74), Q.E.D.

APPENDIX B

PHASE INTERPOLATION WITH A REGULAR ALL-PASS

The phase of a regular all-pass can be made to assume any prescribed set of real values, mod 2π, at given values of $\theta = \theta_1, \theta_2, \ldots, \theta_\ell$. However, the degree of this all-pass can far exceed the integer ℓ unless the phases and the angles θ_i, $i = 1 \rightarrow \ell$, are interrelated in a special way. In this appendix we investigate the possibility of designing a degree-ℓ regular all-pass to interpolate at precisely ℓ points. This of course represents the most economic general situation.

Any real rational regular all-pass $d_\ell(z)$ of degree $\ell \geq 0$ has the structure

$$d_\ell(z) = \epsilon \cdot \frac{\tilde{E}_\ell(z)}{E_\ell(z)} = \epsilon \cdot \frac{z^\ell E_{\ell*}(z)}{E_\ell(z)} \tag{B1}$$

where $\epsilon = \pm 1$ and

$$E_\ell(z) = 1 + f_1 z + \ldots + f_{\ell-1} z^{\ell-1} + f_\ell z^\ell \tag{B2}$$

is a real strict-Hurwitz polynomial with lowest-order coefficient normalized to unity and $f_\ell \neq 0$.

Since $d_\ell(1) = \epsilon$, the problem of most economic phase interpolation may be posed as follows: Let $d_\ell(e^{j\theta}) = \exp[j\psi_\ell(\theta)]$. Then, $\psi_\ell(0) = \frac{1-\epsilon}{2}\pi$ and

$$\psi_\ell(\theta) = \psi_\ell(0) + \arg \frac{\tilde{E}_\ell(e^{j\theta})}{E_\ell(e^{j\theta})} \quad . \tag{B3}$$

Given the two sequences

$$0 = \theta_0 < \theta_1 < \ldots < \theta_\ell < \pi \tag{B4}$$

and

$$0 = \alpha_0, \alpha_1, \ldots, \alpha_\ell \quad , \tag{B5}$$

where $\alpha_k = \psi_\ell(\theta_k) - \psi_\ell(0)$, find $E_\ell(z)$ such that

$$\frac{\tilde{E}_\ell(e^{j\theta_k})}{E_\ell(e^{j\theta_k})} = e^{j\alpha_k}\,, \quad k=0\to\ell \quad . \tag{B6}$$

The original recursive p-plane solution to this problem is due to Rhodes [7, 13] and will repay careful study. Our own z-plane development of his approach places less emphasis on synthesis-theoretic ideas and has a somewhat different motivation.

For every $r = 0\to\ell$, let

$$E_r(g) = 1 + \sum_{k=1}^{r} f_k^{(r)} z^k \tag{B7}$$

denote the real strict Hurwitz polynomial of degree r which interpolates correctly at $\theta = \theta_0, \theta_1, \ldots, \theta_r$. Then, according to (B6) the difference

$$\frac{\tilde{E}_{r+1}}{E_{r+1}} - \frac{\tilde{E}_r}{E_r} = \epsilon(d_{r+1} - d_r) \tag{B8}$$

must vanish at z=1 and $z=e^{\pm j\theta_i}$, $i = 1\to r$. Thus, invoking reality,

$$\frac{\tilde{E}_{r+1}(z)}{E_{r+1}(z)} - \frac{\tilde{E}_r(z)}{E_r(z)} = \frac{\eta_r(1-z)\cdot\prod_{i=1}^{r}(z^2-2z\cos\theta_i+1)}{E_{r+1}(z)E_r(z)} \quad , \tag{B9}$$

η_r a constant. Specifically, by setting z=0 it is immediately verified that

$$\eta_r = f_{r+1}^{(r+1)} - f_r^{(r)} \quad . \tag{B10}$$

Clearing fractions,

$$\tilde{E}_{r+1}(z)E_r(z) - E_{r+1}(z)\tilde{E}_r(z) = \eta_r(1-z)\cdot\prod_{i=1}^{r}(z^2-2z\cos\theta_i+1) \tag{B11}$$

so that

$$\frac{\tilde{E}_{r+2}E_{r+1} - E_{r+2}\tilde{E}_{r+1}}{\tilde{E}_{r+1}E_r - E_{r+1}\tilde{E}_r} = \frac{\eta_{r+1}}{\eta_r}(z^2 - 2z\cos\theta_{r+1}+1) \equiv \Lambda_{r+1}(z) \ . \tag{B12}$$

A simple rearrangement now yields

$$E_{r+1}(\tilde{E}_{r+2} + \Lambda_{r+1}\tilde{E}_r) = \tilde{E}_{r+1}(E_{r+2} + \Lambda_{r+1}E_r) \quad . \tag{B13}$$

Since $E_{r+1}(z)$ is strict Hurwitz, it is relatively prime to $\tilde{E}_{r+1}(z)$ and must therefore divide $E_{r+2}(z) + \Lambda_{r+1}(z)E_r(z)$ without remainder. Clearly, the quotient $q(z)$ is at most of degree one and self-inversive; i.e., $q(z) = \tilde{q}(z)$. Consequently,

$$q(z) = \xi_{r+1}(1+z) \quad , \tag{B14}$$

ξ_{r+1} a real constant, and

$$E_{r+2}(z) + \frac{\eta_{r+1}}{\eta_r}(z^2 - 2z\cos\theta_{r+1} + 1)E_r(z) = \xi_{r+1}(1+z)E_{r+1}(z) \; . \tag{B15}$$

Obviously,

$$\xi_{r+1} = 1 + \frac{\eta_{r+1}}{\eta_r} = \frac{f_{r+2}^{(r+2)} - f_r^{(r)}}{f_{r+1}^{(r+1)} - f_r^{(r)}} \tag{B16}$$

and (B15) takes the final convenient form,

$$E_{r+2}(z) = \xi_{r+1}(1+z)E_{r+1}(z) + (1-\xi_{r+1})(z^2 - 2z\cos\theta_{r+1} + 1)E_r(z) \; , \tag{B17}$$

$$E_0(z) = 1, \quad E_1(z) = 1 + \frac{\sin\frac{1}{2}(\theta_1 - \alpha_1)}{\sin\frac{1}{2}(\theta_1 + \alpha_1)} \cdot z \tag{B18}$$

$$r = 0 \to \ell - 2 \quad .$$

Clearly,

$$f_1^{(1)} = \frac{\sin\frac{1}{2}(\theta_1 - \alpha_1)}{\sin\frac{1}{2}(\theta_1 + \alpha_1)} \tag{B19}$$

and $E_1(z)$ is strict Hurwitz iff

$$|f_1^{(1)}| < 1 \quad . \tag{B20}$$

This granted, we now prove that the remaining polynomials are also strict-Hurwitz provided

$$0 < \xi_r < 1 \ , \quad r = 1 \rightarrow \ell - 1 \quad . \tag{B21}$$

The proof is inductive in character and divides naturally into several parts.

1) From (B17), by direct calculation and iteration,

$$\tilde{E}_{r+2}E_{r+1} - E_{r+2}\tilde{E}_{r+1} = (\xi_{r+1}-1)(z^2 - 2z\cos\theta_{r+1}+1)(\tilde{E}_{r+1}E_r - E_{r+1}\tilde{E}_r) \tag{B22}$$

$$= \eta_{r+1}(1-z) \cdot \prod_{i=1}^{r+1} (z^2 - 2z\cos\theta_i + 1) \tag{B23}$$

where

$$\eta_{r+1} = (f_1^{(1)} - 1) \cdot \prod_{i=1}^{r+1} (\xi_i - 1) \neq 0 \quad . \tag{B24}$$

Thus, $E_k(z)$ has <u>exact</u> degree k, $k = 0 \rightarrow \ell$.

2) Let us assume that all $E_k(z)$, $k = 1 \rightarrow r+1$, are strict-Hurwitz and let us write $E_{r+2}(z) = E_{r+2}(z, \xi_{r+1})$ to stress its dependence on the choice of parameter ξ_{r+1}.

Clearly, as is seen from (B17), $E_{r+2}(z, 0)$ has two distinct zeros

$$z_{1,2} = \exp[\pm j\theta_{r+1}] \tag{B25}$$

on $|z| = 1$, while the remaining r lie in $|z| > 1$. As ξ_{r+1} is increased, z_1 and z_2 either slide together on $|z| = 1$, or move as a pair into $|z| < 1$, or penetrate jointly the region $|z| > 1$. In the first case, there exists ξ_{r+1}, $0 < \xi_{r+1} < 1$, such that $E_{r+2}(z, \xi_{r+1})$ has a zero $z = e^{j\theta}$ of unit modulus.

Since $E_{r+2}(z, 1)$ has the zero $z=-1$ and r+1 zeros in $|z| > 1$, it is obvious that in case two, at least one root locus must leave the interior of the unit circle. Hence, again there exists a ξ_{r+1}, $0 < \xi_{r+1} < 1$, such that $E_{r+2}(z, \xi_{r+1})$ has a zero $z = e^{j\theta}$ of unit modulus.

Consider case three and suppose that for some ξ, $0 < \xi < 1$, $E_{r+2}(z, \xi)$ is not strict-Hurwitz. By continuity, this can only be realized if some root

locus located in $|z| > 1$ crosses the boundary $|z| = 1$ for some ξ_{r+1}, $0 < \xi_{r+1} \leq \xi$.[1] In summary, the assumption that $E_{r+2}(z, \xi)$ is not strict-Hurwitz for some ξ in $0 < \xi < 1$, necessarily implies the existence of a ξ_{r+1}, $0 < \xi_{r+1} < 1$, for which $E_{r+2}(z, \xi_{r+1})$ has a zero $z = e^{j\theta}$ of unit modulus. This we now show is impossible.

3) If $z = e^{j\theta}$ is a zero of $E_{r+2}(z, \xi_{r+1})$, it is also a zero of $\tilde{E}_{r+2}(z, \xi_{r+1})$, and according to (B23) and (B24), either $\theta = 0$ or it coincides with some θ_i, $i = 1 \to r+1$.

If $\theta = 0$, the zero is at z=1 and (B17) yields

$$0 = \xi_{r+1} E_{r+1}(1) + (1-\xi_{r+1})(1-\cos\theta_{r+1})E_r(1) \quad . \tag{B26}$$

However, $E_r(0) = E_{r+1}(0) = 1$ so that $E_r(1) > 0$ and $E_{r+1}(1) > 0$. (If, say, $E_r(1) < 0$, then $E_r(z)$ must vanish for some z in $0 < z < 1$, a contradiction.) Consequently, the right-hand side of (B26) is a sum of positive terms and we have an inconsistency.

If $\theta = \theta_{r+1}$, $E_{r+2}(z, \xi_{r+1})$ has the zero

$$z = \exp[j\theta_{r+1}]$$

and (B17) gives

$$0 = \xi_{r+1} \cdot \cos\frac{\theta_{r+1}}{2} \cdot E_{r+1}(e^{j\theta_{r+1}}) \quad . \tag{B27}$$

But $\cos(\theta_{r+1}/2) \neq 0$ because $0 < \frac{\theta_{r+1}}{2} < \frac{\pi}{2}$ and (B27) is therefore invalid.

Lastly, suppose that for some i, $i = 1 \to r$, $\theta = \theta_i$. Then, from (B17),

$$0 = \xi_{r+1}(1+e^{-j\theta_i})E_{r+1}(e^{j\theta_i}) + 2(1-\xi_{r+1})(\cos\theta_i - \cos\theta_{r+1})E_r(e^{j\theta_i}) \quad . \tag{B28}$$

By construction and the fact that $i \leq r$,

$$\arg\frac{\tilde{E}_{r+1}(e^{j\theta_i})}{E_{r+1}(e^{j\theta_i})} = \arg\frac{\tilde{E}_r(e^{j\theta_i})}{E_r(e^{j\theta_i})} \tag{B29}$$

[1] The coefficient of the highest power of z in $E_{r+2}(z, \xi_{r+1})$ is unequal to zero for all ξ_{r+1} in $0 \leq \xi_{r+1} \leq 1$. This means that no root locus can pass from the exterior to the interior of the unit circle, or vice-versa, by escaping through the point at infinity.

and since for any k

$$\arg \tilde{E}_k(e^{j\theta}) = k\theta - \arg E_k(e^{j\theta}) \quad , \tag{B30}$$

we deduce from (B29) that

$$\arg E_{r+1}(e^{j\theta_i}) = \frac{\theta_i}{2} + \arg E_r(e^{j\theta_i}) \quad . \tag{B31}$$

With the aid of (B31), Eq. (B28) is now reduced to the purely real form,

$$0 = \xi_{r+1} \cos \frac{\theta_i}{2} \cdot \left| E_{r+1}(e^{j\theta_i}) \right| + (1 - \xi_{r+1})(\cos \theta_i - \cos \theta_{r+1}) \cdot \left| E_r(e^{j\theta_i}) \right| \ . \tag{B32}$$

But a moments thought reveals that we are once again faced with a contradiction because $0 < \theta_i < \theta_{r+1} < \pi$ implies that,

$$\cos \theta_i - \cos \theta_{r+1} > 0 \quad . \tag{B33}$$

Thus, it has finally been established that every polynomial $E_r(z)$, $r = 0 \rightarrow \ell$, is strict-Hurwitz, Q.E.D.

The restrictions

$$0 < \xi_r < 1 \ , \quad r = 1 \rightarrow \ell - 1 \quad , \tag{B34}$$

impose constraints on the admissible interpolatory pairs (θ_i, α_i), $i = 0 \rightarrow \ell$. These constraints reflect the well-known fact that the frequency variation of a regular all-pass is not entirely arbitrary.

For example, as a consequence of the strict-Hurwitz character of $E_k(z)$ we can easily prove that

$$\frac{d}{d\theta} \arg \frac{\tilde{E}_k(e^{j\theta})}{E_k(e^{j\theta})} = \frac{d\psi_k(\theta)}{d\theta} \geq 0 \tag{B35}$$

and

$$\psi_k(\pi) - \psi_k(0) = k\pi \quad . \tag{B36}$$

Thus, the α's must at least form a monotone-increasing sequence:

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_\ell < \ell\pi \quad . \tag{B37}$$

Unfortunately, (B37) does not always guarantee (B34) and the general situation is rather complicated. Nevertheless, because of their great practical importance, we shall carry out the designs for a single and doubly-tuned FEE in detail. Recall that according to Eqs. (104), (108) and (109),

$$K_{FE}(\theta) = K_{ME}(\theta) \cdot \frac{1-\mu^2}{1-2\mu\cos\varphi(\theta)+\mu^2} \tag{B38}$$

where

$$\varphi(\theta) = \psi(\theta) + \beta_n(\theta) + \theta \quad , \tag{B39}$$

and $\psi(\theta)$, $\beta_n(\theta)$ are the phase angles of $d(e^{j\theta})$ and $U_n(e^{j\theta})$, respectively.

Design of a Singly-Tuned FEE. Suppose that maximum gain

$$\frac{1+\mu}{1-\mu} \tag{B40}$$

is desired at some single suspected peak observed in $K_{ME}(\theta)$, say, at location $\theta = \theta_1$. Clearly, we must have $\varphi(\theta_1) = 0$, mod 2π, and this we attempt to achieve with a regular all-pass $d_1(z)$ of degree one.

Let

$$\gamma_1 = \beta_n(\theta_1) + \theta_1, \text{ mod } 2\pi \ , \tag{B41}$$

so that $0 \leq \gamma_1 < 2\pi$ and let[2]

$$\epsilon = \text{sign}(\gamma_1 - \pi) \quad . \tag{B42}$$

If $\epsilon = 0$ we pick $d_1(z) = -1$. Otherwise, if $\epsilon = +1$ we choose $\arg \psi_1(\theta_1) = \alpha_1 = 2\pi - \gamma_1$ and if $\epsilon = -1$ we set $\psi_1(\theta_1) = \pi + \alpha_1 = 2\pi - \gamma_1$. Thus, from (B18),

---

(2)

$$\text{sign } x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$E_1(z) = 1 + \frac{\sin\frac{1}{2}(\theta_1+\gamma_1+\frac{1-\epsilon}{2}\pi)}{\sin\frac{1}{2}(\theta_1+\gamma_1+\frac{1-\epsilon}{2}\pi)} \cdot z \tag{B43}$$

and

$$d_1(z) = \epsilon \cdot \frac{\tilde{E}_1(z)}{E_1(z)} \quad . \tag{B44}$$

It is easily confirmed that

$$f_1^{(1)} = \frac{\sin\frac{1}{2}(\theta_1+\gamma_1+\frac{1-\epsilon}{2}\pi)}{\sin\frac{1}{2}(\theta_1-\gamma_1+\frac{1-\epsilon}{2}\pi)} \tag{B45}$$

has magnitude less than one, as required.

<u>Design of a Doubly-Tuned FEE.</u> Let us now attempt to achieve maximum gain at two locations θ_1 and θ_2, $0 < \theta_1 < \theta_2 < \pi$, with a regular all-pass $d_2(z)$ of <u>degree two</u>. Let

$$\gamma_i = \beta_n(\theta_i) + \theta_i \,, \quad \text{mod } 2\pi \,, \tag{B46}$$

$$i = 1, 2 \,.$$

Since $\psi_2(\theta_1) - \psi_2(0) = \alpha_1$ and $\psi_2(\theta_2) - \psi_2(0) = \alpha_2$ and since the phase angle $\psi_2(\theta) - \psi_2(0)$ of the regular all-pass $\tilde{E}_2(z)/E_2(z)$ must be monotone-increasing, it is necessary that $\alpha_1 < \alpha_2$.

This latter observation leads to the following assignment.

1) If $\gamma_1 \geq \gamma_2$; $\epsilon = 1$, $\alpha_i = 2\pi - \gamma_i$, $i = 1, 2$.

2) If $\gamma_1 < \gamma_2$, $\gamma_1 \leq \pi$, $\gamma_2 \geq \pi$; $\epsilon = -1$, $\alpha_1 = \pi - \gamma_1$, $\alpha_2 = 3\pi - \gamma_2$.

3) Any other case is impossible with a degree-two regular all-pass.[3]

In cases 1) and 2), $E_1(z)$ is given by Eq. (B43).

To calculate $E_2(z)$ we can use (B17) with $r = 0$:

$$E_2(z) = \xi_1(1+z)E_1(z) + (1-\xi_1)(z^2 - 2z\cos\theta_1 + 1) \quad . \tag{B47}$$

---

[3] The maximum phase increment attainable with $d_2(z)$ over $0 \leq \theta \leq \pi$ equals 2π and occurs at $\theta = \pi$.

Of course, ξ_1 is unknown and must be determined from the interpolatory condition,

$$\arg \frac{\tilde{E}_2(e^{j\theta_2})}{E_2(e^{j\theta_2})} = \alpha_2 \quad . \tag{B48}$$

Clearly,

$$E_2(e^{j\theta_2}) = \xi_1(1+e^{j\theta_2})E_1(e^{j\theta_2}) + 2(1-\xi_1)\, e^{j\theta_2}(\cos\theta_2 - \cos\theta_1)$$

and we find that

$$e^{j\alpha_2} = \frac{\bar{y}+x}{y+x} \tag{B49}$$

where

$$y = E_1(e^{j\theta_2})e^{-j\theta_2/2} \quad , \quad x = \frac{1-\xi_1}{\xi_1} \cdot \frac{(\cos\theta_2 - \cos\theta_1)}{\cos\frac{\theta_2}{2}} \quad . \tag{B50}$$

Finally, solving for ξ_1 with the help of (B43) we get (algebra omitted),

$$\xi_1^{-1} = 1 + \frac{\cos\frac{\theta_2}{2}}{\cos\theta_1 - \cos\theta_2} \cdot \frac{\sin\frac{1}{2}(\theta_2+\gamma_2+\frac{1-\epsilon}{2}\pi) - f_1^{(1)} \sin\frac{1}{2}(\theta_2-\gamma_2-\frac{1-\epsilon}{2}\pi)}{\sin\frac{1}{2}(\gamma_2+\frac{1-\epsilon}{2}\pi)} \tag{B51}$$

Let us check to see if $0 < \xi_1 < 1$ when $\epsilon = 1$. For this it suffices that

$$\frac{\sin\frac{1}{2}(\theta_2+\gamma_2)}{\sin\frac{1}{2}(\theta_2-\gamma_2)} - \frac{\sin\frac{1}{2}(\theta_1+\gamma_1)}{\sin\frac{1}{2}(\theta_1-\gamma_1)} \lesseqgtr 0 \, , \tag{B52}$$

depending on whether

$$\theta_2 - \gamma_2 \gtrless 0 \quad , \tag{B53}$$

respectively. These inequalities are not automatic and we shall attempt to clarify this matter in Part II.

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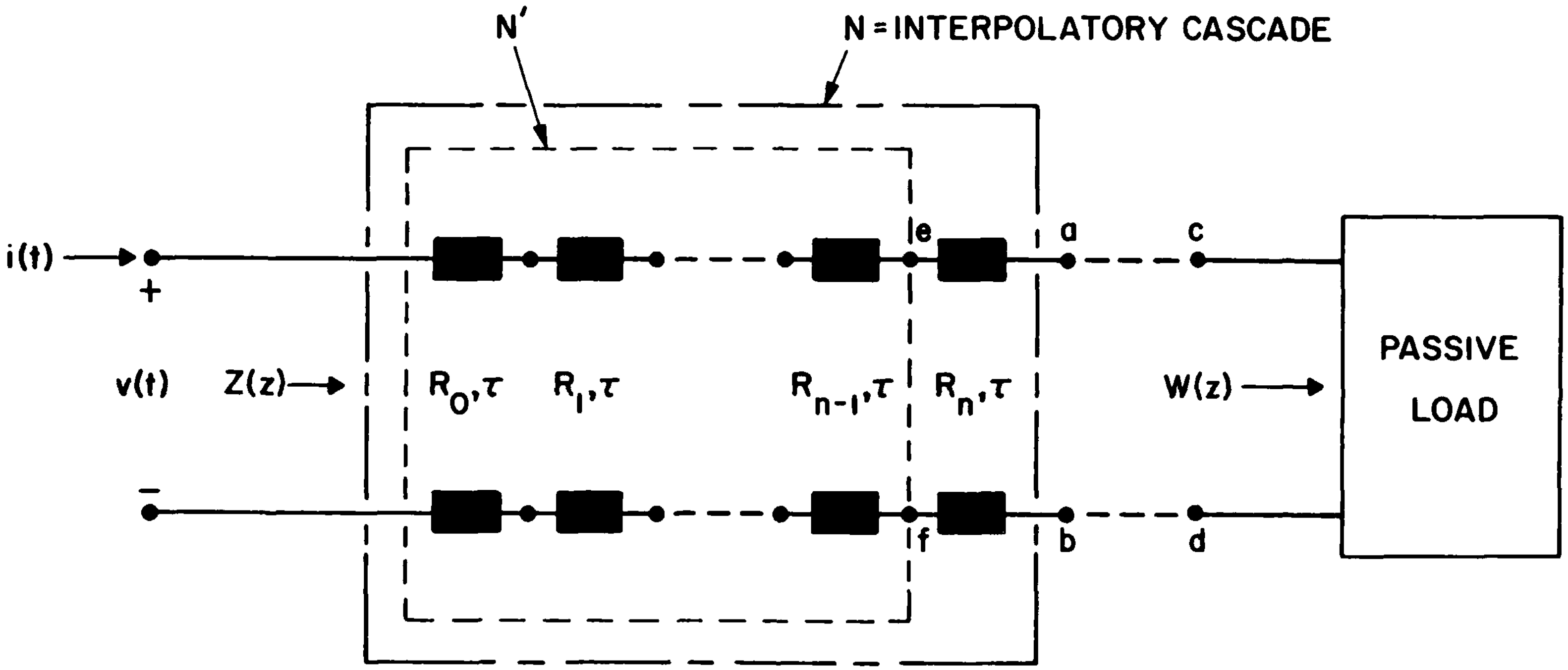
$a_{0t} = R_0 = C(0)\ ;\ z = e^{-2p\tau}\ ;\ p = \sigma + j\omega$



W(z) = DRIVING-POINT IMPEDANCE OF PASSIVE LOAD.

Z(z) = INPUT IMPEDANCE OF CASCADE WITH TERMINALS cd CLOSED ON ab.

$$\text{FOR } i(t) = \delta(t)\ ,\ v(t) = C(0)\delta(t) + \sum_{r=1}^{n} 2C(r)\delta(t - 2r\tau) + \cdots$$

Fig. 1

PART II

A DIGITAL IMPLEMENTATION

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1. Introduction

The solution to the problem of the spectral estimation as proposed by Professor Youla of Brooklyn Polytechnic offered many attractive features. One of them is the ability to tune the spectrum at desired frequencies. The technique, called Flat Echo Estimator (FEE) is described in a paper by Professor Youla.[1] A computer program, described in the following, was developed to implement this technique in FORTRAN IV on MULTICS. The brief description of the algorithm is followed by the block diagram, description of various subroutines and a listing of the FORTRAN program itself. A user documentation to run this program is also provided in this report.

2. Algorithm

Youla has shown that the maximum entropy spectral estimates $K_{ME}(\theta)$, for the finite sequence of error-free covariance samples $C(1)$, $C(2)$,..., $C(n+1)$ given by

$$K_{ME}(\theta) = \frac{1}{|P_n(e^{j\theta})|^2} \tag{1}$$

can be viewed as the real part of the driving-point impedence of a cascade of (n+1) commensurate lines terminated on a load $W(z)(=R_n)$. The resistance R_n is the same as the characteristic impedance of the last line in the cascade. He showed that any other positive-real load termination will also provide an admissible spectral estimate for the same covariance sample sequence. If the load network is an all-pass network with reflection factor

$$\rho(e^{j\theta}) = \mu d(e^{j\theta}) = \mu \exp[j\psi(\theta)] \quad , \tag{2}$$

the spectral estimates provided will be $K_{FE}(\theta)$ and could be derived to be

$$K_{FE}(\theta) = K_{ME}(\theta) \cdot \frac{1-\mu^2}{1-2\mu\cos\psi(\theta) + \mu^2} \tag{3}$$

Where

$$\zeta(\theta) = \psi(\theta) + \beta_n(\theta) + \theta$$

The phase factor $\beta_n(\theta)$ is given by the expression

$$\exp[j\,\beta_n(\theta)] = \frac{\tilde{P}_n(z)}{P_n(z)} = \frac{z^n P_n(1/z)}{P_n(z)} \quad , z = e^{j\theta} \tag{4}$$

The polynomial $P_n(z)$ here is the same as that in the equation (1) and is given by

$$P_n(z) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} C(1) & C(2) & \cdots & C(n+1) \\ C(2) & C(1) & \cdots & C(n) \\ \cdots & \cdots & \cdots & \cdots \\ C(n) & C(n-1) & \cdots & C(2) \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix} \tag{5}$$

The determinant Δ_k is given by

$$\Delta_k = \begin{vmatrix} C(1) & C(2) & \cdots & C(k+1) \\ C(2) & C(1) & \cdots & C(k) \\ \cdots & \cdots & \cdots & \cdots \\ C(k+1) & C(k) & \cdots & C(1) \end{vmatrix} \tag{6}$$

The factor $G(\theta)$

$$G(\theta) = \frac{1-\mu^2}{1-2\mu\cos\psi(\theta) + \mu^2} \tag{7}$$

in the expression for $K_{FE}(\theta)$ is similar to a tuned amplifier gain factor. The maximum antropy thus can be tuned to provide for peaks at any desired frequency by a proper choice of the all-pass load network's phase characteristics $\psi(\theta)$. The choice of $(0 \leq \mu < 1)$ provides for adjusting the amplitude of the peak. It should by noted that $\mu=0$ reduces K_{FE} to K_{ME}. Professor Youla has demonstrated a link between the variable μ and the numerical robustness of the spectral estimator. The condition $\mu=0$ reduces K_{FE} to K_{ME} and represents the maximum numerically robust choice.

3. Computer Program

The flow chart to compute $K_{FE}(\theta)$ for a given sequence $C(1), C(2), \ldots, C(n+1)$ is shown in Figure 1. The first step to obtain $K_{FE}(\theta)$ is to compute $K_{ME}(\theta)$, the spectral estimates by the maximum entropy method, then design an all-pass network and finally compute $G(\theta)$. The multiplication of $G(\theta)$ and $K_{ME}(\theta)$ will provide the value of $K_{FE}(\theta)$. It should be noted that we don't have to find the actual element values of the all-pass network.

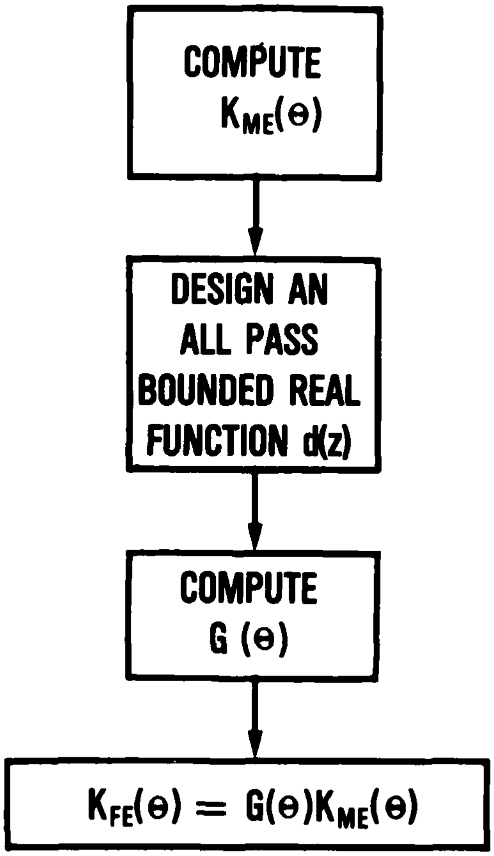

FIG. 1: FLOW CHART TO ESTIMATE SPECTRUM BY FLAT ECHO ESTIMATOR METHOD

Rather, the computation of an appropriate all-pass bounded-real function d(z) (see equation 2) will be enough.

3.1 Computation of $K_{ME(\theta)}$

The $K_{ME}(\theta)$, the spectral estimate by maximum entropy method is computed as follows. First a matrix t_n is formed from the covariance coefficients C(1), C(2), ... C(npts) as

$$t_n = \begin{bmatrix} C(1) & C(2) & \ldots & C(npts) \\ C(2) & C(1) & \ldots & C(npts-1) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ C(npts-1) & \ldots & \ldots & C(2) \end{bmatrix} \qquad (8)$$

Then the coefficients of a polynomial $Q_n(z)$ (which is the same as $P_n(z)$ except for the constant multiplying factor ($= 1/\sqrt{\Delta_n \Delta_{n-1}}$) in eqn. (5)) are obtained as the determinant of a matrix t_{n_i} formed by deleting an appropriate column i of matrix t_n.

The coefficient of Z^n for example, is found as the determinant of a matrix t_{n1} obtained by deleting the first column of t_n. The reason for doing so can be seen by noting that $Q_n(z)$ is the determinant of the matrix formed by t_n appended by a vector $[z^n, z^{n-1}, \ldots, 1]$ as its last row, i.e.,

$$\sqrt{\Delta_n \Delta_{n-1}}\, P_n(z) = Q_n(z) = \begin{vmatrix} t_{n_i} & t_{n\,2} & \ldots & t_{n\,npts} \\ z^{npts-1} & z^{npts-2} & \ldots & 1 \end{vmatrix}$$

$$= \sum_{m=1}^{npts} (-1)^m \, |t_{nm}| \, z^{npts-m} \qquad (9)$$

The computation of det $[t_{nm}]$ is performed by calling the subroutine pme cof.

A constant multiplying factor C_{mult} is obtained as the product of det n ($=\Delta_n$) and Δ_{n-1} . It can be seen that Δ_{n-1} is same as the constant term in $Q_n(z)$. The determinant Δ_n is obtained in the do loop to compute coefficients of $Q_n(z)$.

Finally, the coefficients "coef" of polynomial $P_n(z)$ are obtained from the coefficients of $Q_n(z)$ by dividing each one of them by $1/\sqrt{Cmult}$.

The spectrum $K_{ME}(\theta)$ may be obtained at this point as $\dfrac{1}{|P_n(e^{j\theta})|^2}$ if so desired. The flow chart to compute $K_{ME}(\theta)$ is shown in Figure 2.

3.2 Computation of FEE Spectral Estimates

Our goal is to obtain $K_{FE}(\theta)$. Therefore, instead of simply obtaining $K_{ME}(\theta)$ at the above step, a subroutine spectral estimator is called. This subroutine first queries if only K_{ME} is desired or K_{FE} is desired. For K_{FE}, it asks further questions as to singly tuned or doubly tuned, the value of these frequencies (in radians) and the maximum robustness factor AMU. It then calls an appropriate subroutine design-all-pass or design-all-pass-2 to determine a first order or second order all-pass reflection factor. The subroutine "response" is used to compute the spectrum nres points for θ between 0.0 and "thmax". Upon return from spectral-estimator, the results can be written in an output file to provide for readouts and/or plotted on Tek screen by calling plot routines appropriately. The flow chart of the computer programs is shown in Figure 3.

4. Subroutine Description

The computer program has been broken down into very many smaller programs called subroutines. A list of subroutines used herein is shown with their interconnection in Figure 4. A brief description of various subroutines is provided in the following:

4.1 Subroutine "Spectral-Estimator"

4.1.1 Purpose: The purpose of this subroutine is to query the user if FEE or MEM spectral estimates are required. In case FEE is desired, it queries the user about the robustness factor AMU and the tuned frequencies. It then designs an appropriate all-pass reflection coefficient (first order for singly tuned by calling "design-all-pass" or second order for doubly tuned by calling "design-all-pass-2". The queries for FEE parameters and the design of all-pass network is skipped if only MEM is desired. It then calls "response" to compute the spectral estimates using equation (3). The flow diagram of this subroutine is shown in Figure 5.

4.1.2 List of Variables: Following is the list of variables used in "Spectral Estimator"

nfee - integer variable = 0 for MEM

= 1 for singly tuned FEE

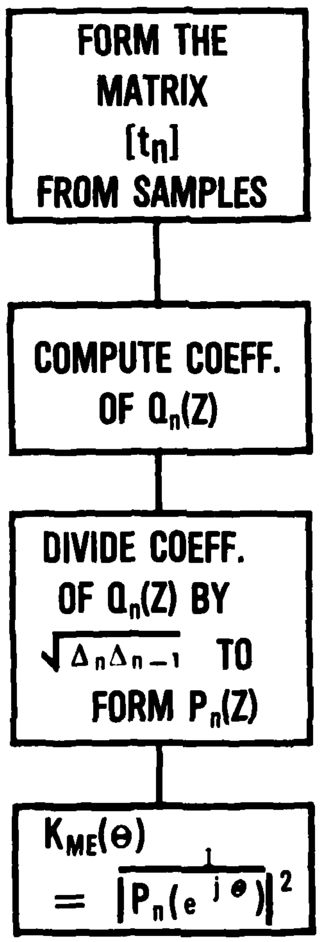

FIG. 2: FLOW CHART TO COMPUTE $K_{ME}(\Theta)$

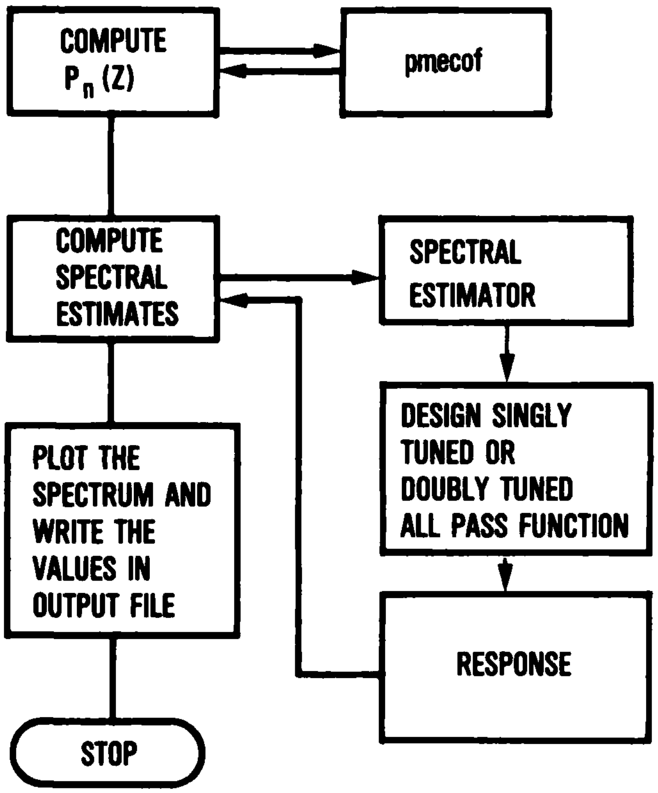

FIG. 3: OVERALL PROGRAM–FLOW DIAGRAM

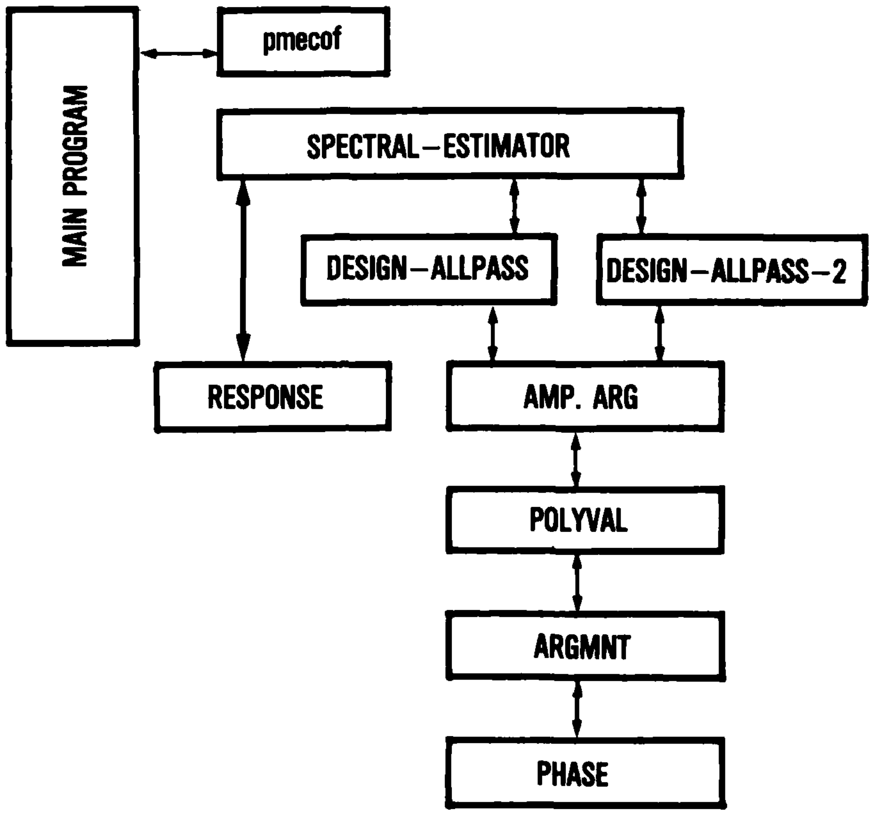

FIG. 4: INTERCONNECTION OF VARIOUS SUBROUTINES

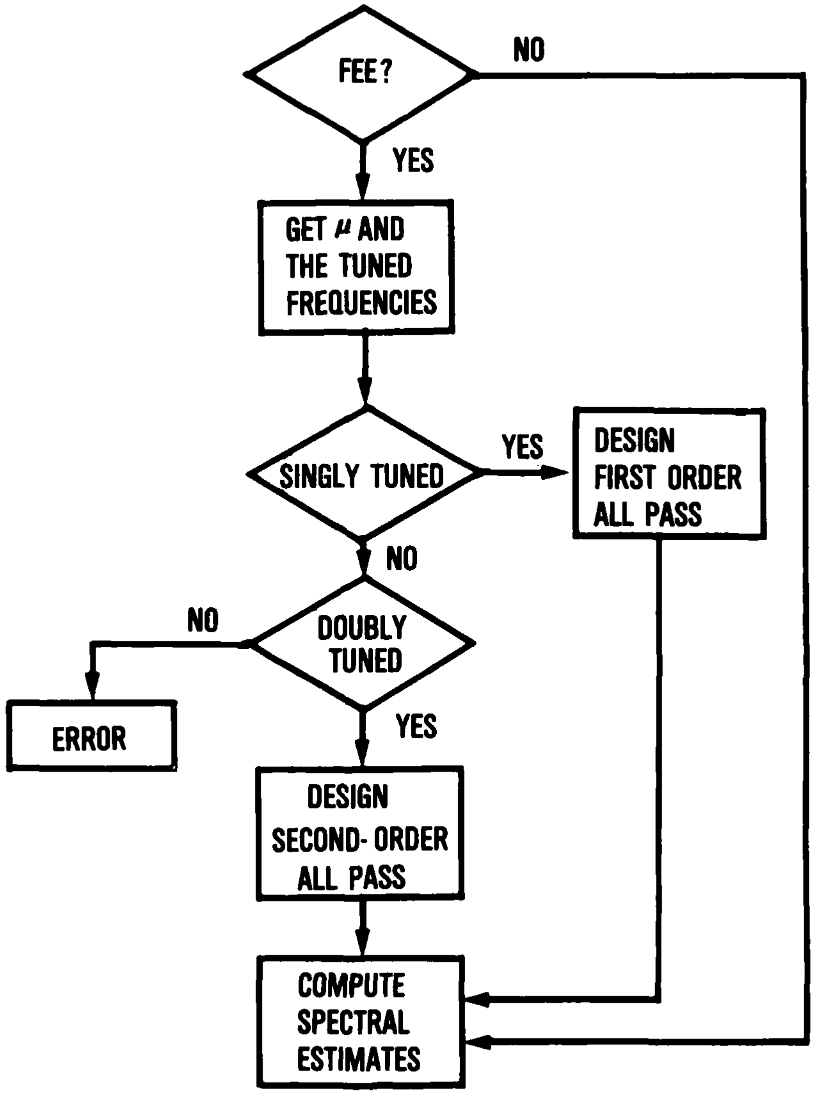

FIG. 5: FLOW CHART - SUBROUTINE SPECTRAL ESTIMATOR

= 2 for doubly tuned FEE

amu - the numerical robustness factor

peak - a vector of frequencies (in radians) at which tuning of the spectrum is desired. The frequencies are arranged in the increasing order. The smallest desired frequency must be peak (1).

nerror - a flag to indicate that all-pass network function cannot be obtained

npts-d - the order of the polynomial E_ℓ in the all-pass reflection factor

$$d(z) = \frac{\tilde{E}_\ell(z)}{E_\ell(z)}$$

npts-d = 2 for singly tuned case
= 3 for doubly tuned case

cf-d - the coefficients of the polynomial

4.2 Subroutine Design all pass

4.2.1 Purpose: This subroutine is to design a first order all-pass network function d(z) whose phase is prescribed at one frequency (peak (1)). This computer program is based on the algorithm provided by Youla in Reference 1. First the phase GAMMA 1 = $[\beta(\theta_1) + \theta_1] \bmod 2\pi$ at θ_1 = peak (1) is computed. The all-pass phase function is to be chosen in such a way that $\psi(\theta_1)$ + GAMMA 1 = [0] mod2π where $\psi(\theta_1)$ is the phase of the all-pass d(z) at peak (1).

4.3 Subroutine amp-arg

4.3.1 Purpose: Given a polynomial R(z)

$$R(z) = a_0 + a_1(z) + a_2z^2 + \ldots + a_nz^n$$

This subroutine computes the amplitude and phase of R(a) for any given "th"

$$z = e^{j(th)}$$

The subroutine polyval is used to compute the complex value of polynomial R(z) and argmnt is used to determine the phase of R(z) between 0 and 2π.

It is frequently desired in this program to evaluate the phase of the all-pass function.

$$d(z) = e^{j\gamma}\frac{\tilde{E}_\ell(z)}{E_\ell(z)} = e^{j\gamma}z^n \frac{E_\ell(1/z)}{E_\ell(z)}$$

$$\begin{aligned} \arg\,[d(z)] &= \gamma + (n-1)\theta + \arg\,[E_\ell(z^*)] - \arg\,[E_\ell(z)] \\ &= \gamma + (n-1)\theta - 2\alpha(\theta) \end{aligned}$$

and γ is some constant phase (0 or 2π)

4.4 Subroutine Polyval

This subroutine computes the value (complex) of a polynomial.

$$R(z) = a_1 + a_2 z + a_3 z^2 + \ldots + a_n z^{n-1}$$

for complex z. The coefficients a_i are real.

4.5 Subroutine Argument

This subroutine calculates the argument of a complex number. The result is between 0 and 2π depending upon the signs of the real and imaginary parts.

4.6 Subroutine "response"

This subroutine calculates the spectral estimates for K_{ME} or K_{FE} using appropriate expressions at nres points for θ ranging from 0 to "thmax". For maximum entropy, K_{ME} is given by

$$fx(i) = K_{ME}(\theta i) = \frac{1}{|P_n(e^{j\theta i})|^2}$$

For K_{FE}, K_{ME} is multiplied by a factor Amult given by equation (7). Most of the variables used in this routine are self-explanatory.

4.7 Subroutine design-all-pass.2:

4.7.1 Design of a doubly tuned FEE

The purpose of this routine is to design an all-pass network function whose phase has been prescribed at two frequencies. Except for some minor changes, the algorithm used is the same as provided by Youla in Reference(1). The first step in the process is to compute the desired phases Gamma 1 and Gamma 2. (eq. B(46) of Reference (1)) at peak (1) and peak (2). Then the phases alpha 1 and alpha 2 are computed depending upon the relative values of Gamma 1 and Gamma 2.

We know that (see eq. B48 of Reference (1))

$$\arg \left. \frac{E_2(z)}{E_2(z)} \right|_{z=e^{j\theta_2}=\alpha_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.7.1)$$

$$\frac{z^2[(E_2(1/z)]}{E_2(z)} = e^{jz\theta_2 - 2\arg[E_2(z)]\big|_{z=e^{j\theta_2}}} = e^{j\alpha_2}$$

$$\arg[E_2(z)]\Big|_{z=e^{j\theta_2}} = \theta_2 - \alpha_2/2 \quad , \ldots\ldots\ldots \qquad (4.7.2)$$

But also (see eq. (B-47) of Reference (1))

$$E_2(e^{j\theta_2}) = \xi_1(1+e^{j\theta_2}) \cdot E_1(e^{j\theta_2}) + (1-\xi_1)(e^{2j\theta_2} - 2e^{j\theta_2}\cos\theta_1 + 1)$$

$$= 2\xi_1(\cos\theta_2/2 \cdot e^{j\theta_2/2})E_1(e^{j\theta_2}) + (1-\xi_1)e^{j\theta_2}(2\cos\theta_2 - 2\cos\theta_1)$$

$$= 2\xi_1 \cos{}_2/2\; e^{j\theta_2}(E_1(e^{j\theta_2})\; e^{j\theta_2/2} + \frac{1-\xi_1}{\xi_1} \cdot \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2/2}$$

$$\arg E_2(z)\Big|_{z=e^{j\theta_2}} = \arg[e^{j\theta_2}(y+x)] \ldots\ldots\ldots\ldots\ldots \qquad (4.7.3)$$

$$\text{where } y = E_1(e^{j\theta_2})e^{-j\theta_2/2} \ldots\ldots\ldots\ldots \qquad (4.7.4)$$

$$x = \frac{1-\xi_1}{\xi_1} \cdot \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2/2} \ldots\ldots\ldots \qquad (4.7.5)$$

From eq. (4.7.2)

$$\theta_2 - \alpha_2/2 = \theta_2 + \arg\,(y+x)$$

or $\quad \arg(y+x) = \alpha_2/2 = a$

i.e., $\quad \mathrm{Re}(y+x) = R\cos A$

$\quad \mathrm{Im}(y+x) = R\sin A$

but $\quad E_1(z) = 1 + f_1 z = 1 + f_1 e^{j\theta_2}$

$$y = E_1(z)e^{-j\theta_2/2} = e^{-j\theta_2/2} + f_1 e^{j\theta_2}$$

$$y = \cos\theta_2/2 - j\sin\theta_2/2 + f_1\cos\theta_2/2 + f_1 j\sin\theta_2/2$$

$$y = (1 + f_1)\cos\theta_2/2 + j(f_1-1)\sin\theta_2/2 \qquad (4.7.6)$$

Therefore from eq. (4.7.6)

$$x + (1+f_1)\cos\theta_2/2 = R\cos A \quad \ldots\ldots\ldots\ldots \quad (4.7.7)$$

$$(f_1-1)\sin\theta_2/2 = R\sin A \quad \ldots\ldots\ldots\ldots\ldots \quad (4.7.8)$$

Since f_1 is known (being the coefficient of z in $E_1(z)$ of singly tuned case), R can be determined from eq. (4.7.8). This R is then substituted in eq. (4.7.7) to obtain x .

Once x is known, ξ_1 can be computed as, from eq. (4.7.5)

$$\frac{1-\xi_1}{\xi_1} = \frac{x\cos\theta_2/2}{\cos\theta_2-\cos\theta_1} = B$$

$$\xi_1 = \frac{1}{1+B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.7.9)$$

Then the coefficients of polynomial $E_2(z)$ are determined by eq. B(47), proof in Ref (1), as

$$\begin{aligned} E_2(z) &= \xi_1(1+z)(1+f_1(z)) + (1-\xi_1)(z^2-2z\cos\theta_1+1) \\ &= \xi_1[1+(1+f_1)z + f_1z^2] + (1-\xi_1)(1-2z\cos\theta_1 + z^2) \\ &= 1 + z[\xi_1(1+f_1) - 2\cos\theta_1(1-\xi_1)] + z^2(\xi_1 f_1 + (1-\xi_1)) \end{aligned}$$

4.4.2 List of Variables:

The variables used in this subroutine are fairly self-explanatory.

Vector cf-d = coefficients values of polymial $E_2(Z)$
npts-d = number of coefficients of $E_2(Z)$
(=3 for doubly tuned case)
ar = A of eq. (4.7.6)
br = B of equ. (4.7.9

5. User Instructions

The computer program is implemented in FORTRAN IV and the listing is shown in Appendix A. The program is named as FEE2 on Multics. Before running the program, the user must make sure that the graphics mode for his system is turned on. A typical user session is described below. The system's response or questions are marked as "S" whereas the user response is marked as U. (see figure 7)

S: γ
U: fee2
S: enter two digit file number (note 1)
   enter number of parts to be used
S: enter 0 for mem, 1 for singly tuned and 2 for doubly tuned fee
   (note 2)
U: 0,1, or 2
The following questions are not asked it the user enters 0 in the above response.
S: enter robustness factor - amu
U: any real number μ, $0<\mu<1$ (note 3)
S: enter desired resonant peaks (in radians)
U: any real number between 0 and 1 (note 4)
S: The estimated spectral plots are plotted, the μ and resonant peak values printed and then, the following statement typed: If you wish to make a hard copy, do it now.
   Then type 1, if you wish to plot the phase of the all pass network.
   Else, type in 0.
U: 0 or 1
S: depending upon the user's response to the above question, the phase of the all pass FEE network load is plotted
S: Stop
S: γ

5.1 Notes:

5.1.1 Note 1: The program at this stage is asking you to input two digits XX (=25 in test case) of the file XX on the terminal. The program assumes that the following input data is pre-stored in file XX (file 25, for example).

Input data in file XX:

(i) nres (number of points at which estimated spectral response is to be evaluated in format I4)

(ii) thmax (the maximum value of θ (in radians) up to which spectral estimates are to be computed in format E18,10)

(iii) npts (in format I2, the number of error-free covariance sample data points)

(iv) The covariance values at npts points (in the format E18.10)

For the test case, the data stored in file 25 is shown in fig.6. The MEM and FEE spectral estimates are shown in figures 8 through 11.

5.1.2 Note 2: The program at this point is asking if the maximum entropy spectral estimates or FEE estimates are desired. If the user types in 0 (for MEM), the program does not ask any further questions, and simply plots the MEM spectral estimates on the terminal screen. In the case of FEE (user response of 1 for singly tuned and 2 for doubly tuned), the user has to type in the robustness factor μ and the desired resonant peaks.

5.1.3 Note 3: The robustness factor μ should be a real number, between 0.0 and 1.0. If you type 0.0, the vector will be the same as MEM estimates. The program would bomb if $\mu = 1,0$.

5.1.4 Note 4: The values of resonant peaks in radians that you are asked to input (real numbers) must be 1 for singly tuned and 2 for doubly tuned case. Furthermore, the peaks for doubly tuned case must be in increasing order, i.e. the first input value must be less than the second input value.

5.1.5 Note 5: At this point, the system has computed and plotted the desired spectral estimates. It has also printed the other useful information, such as μ and peak values for the hard copy records. The program can also plot the phase of the all pass function d(Z). It asks the user if he wishes to plot this phase. Since the program shall blank out the screen before plotting this all pass phase function, the user must make a hard copy of spectral estimates before responding to this question.

```
1, p
1024
 6.283184
  2.020000
  1.622227
  1.076796
  0.516477
  0.117178
  0.012131
  0.095929
  0.346192
  0.590948
  0.728907
  0.743116
  0.596789
  0.396138
  0.190008
  0.043111
  0.004463
  0.035287
  0.127353
  0.217394
  0.268148
```

Fig. 6: Data for Test Example (contained in file 25)

```
Setup graphics-table tek 4014
segment Setup-graphics not found.
r 1217 0.055 1.750 50

Setup graphics-table tek 4014
r 1213 0.351 3.906 62

fee2
enter two digit file number
25
enter 0 for mem, 1 for singly and 2 for doubly tuned fee
2
enter robustness factor -: amu
0.5
enter desired resonant peak (in radians)
0.6035
enter desired resonant peak (in radians)
2.502
```

Fig. 7: A typical user session

DOUBLY TUNED FEE

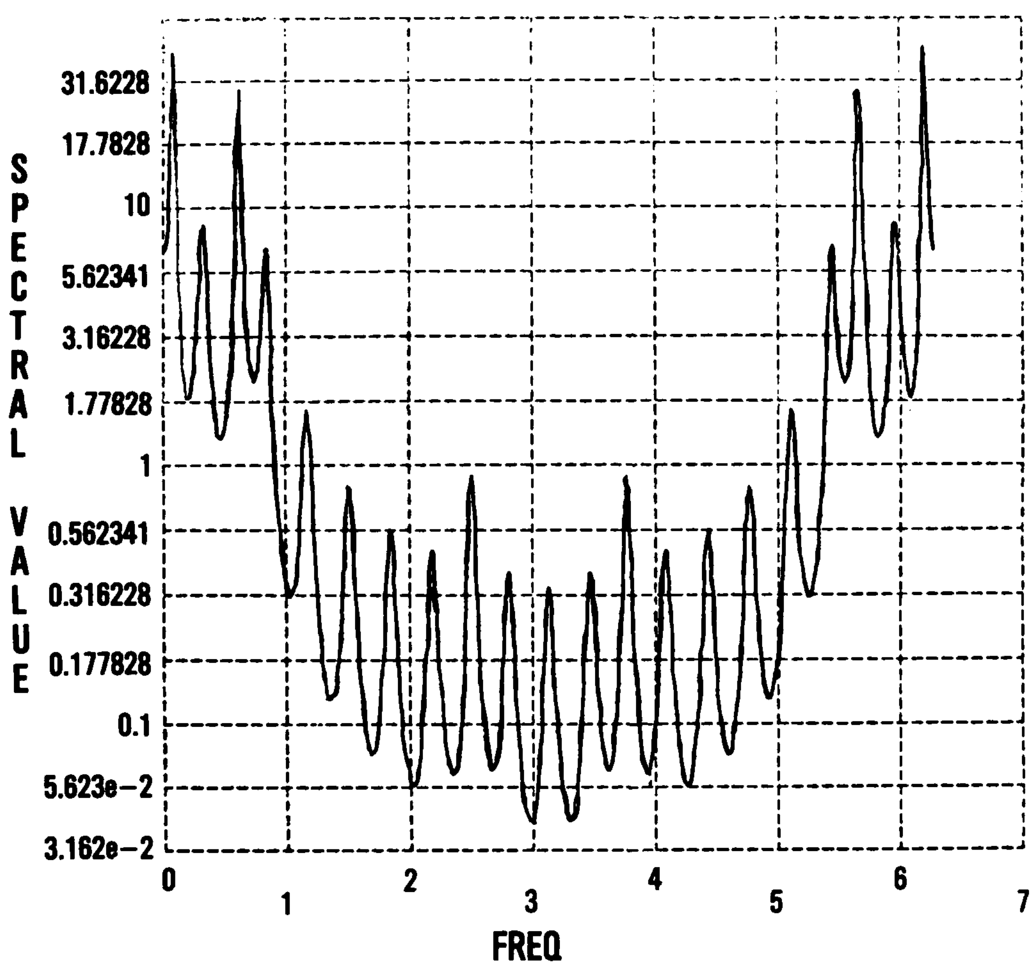

FIG. 8a: SPECTRAL ESTIMATES—DOUBLY TUNED FEE

(SEE FIG. 6 FOR TEST DATA)

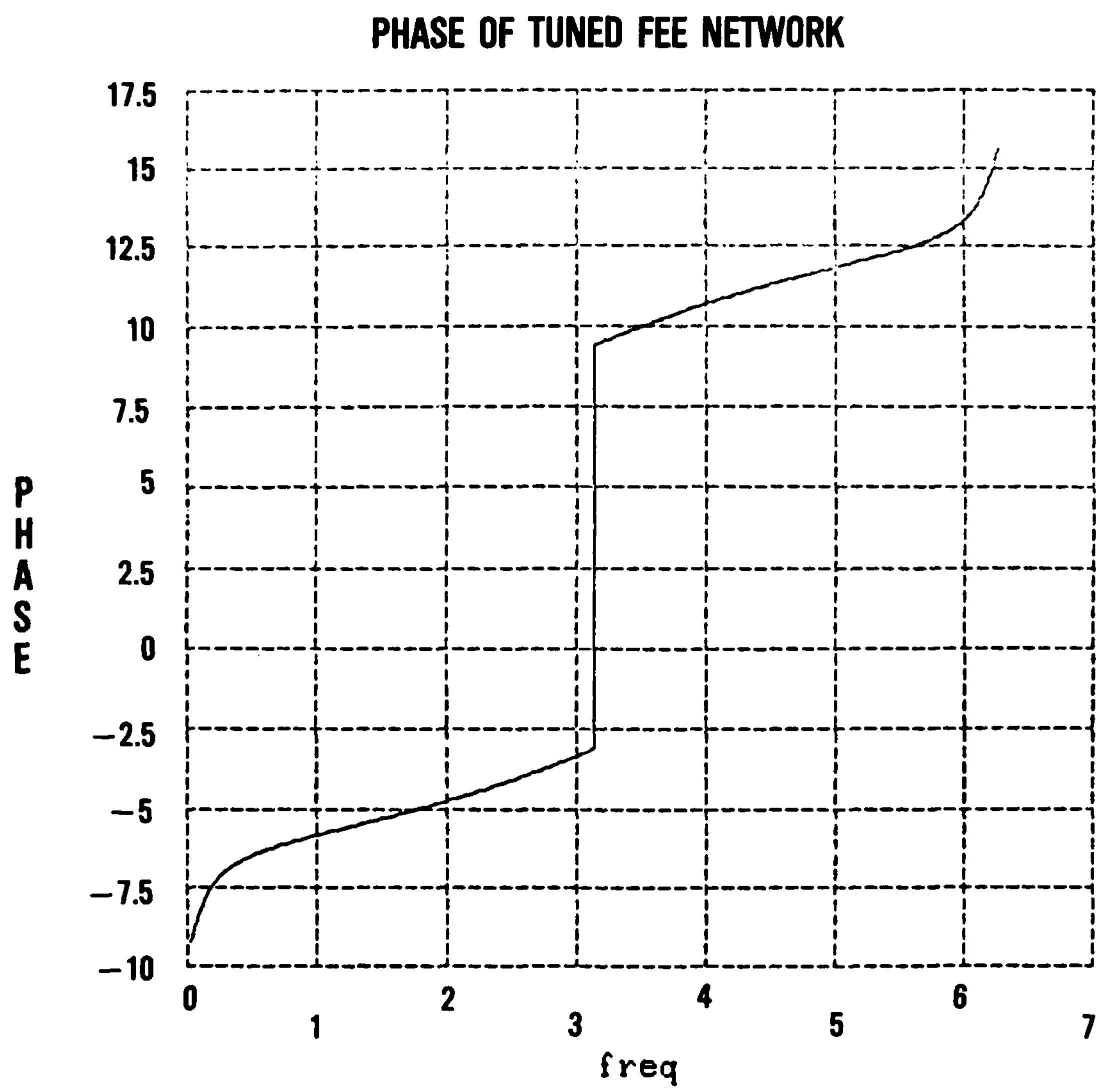

FIG. 8b: PHASE OF THE ALL—PASS LOAD NETWORK FOR DOUBLY TUNED FEE

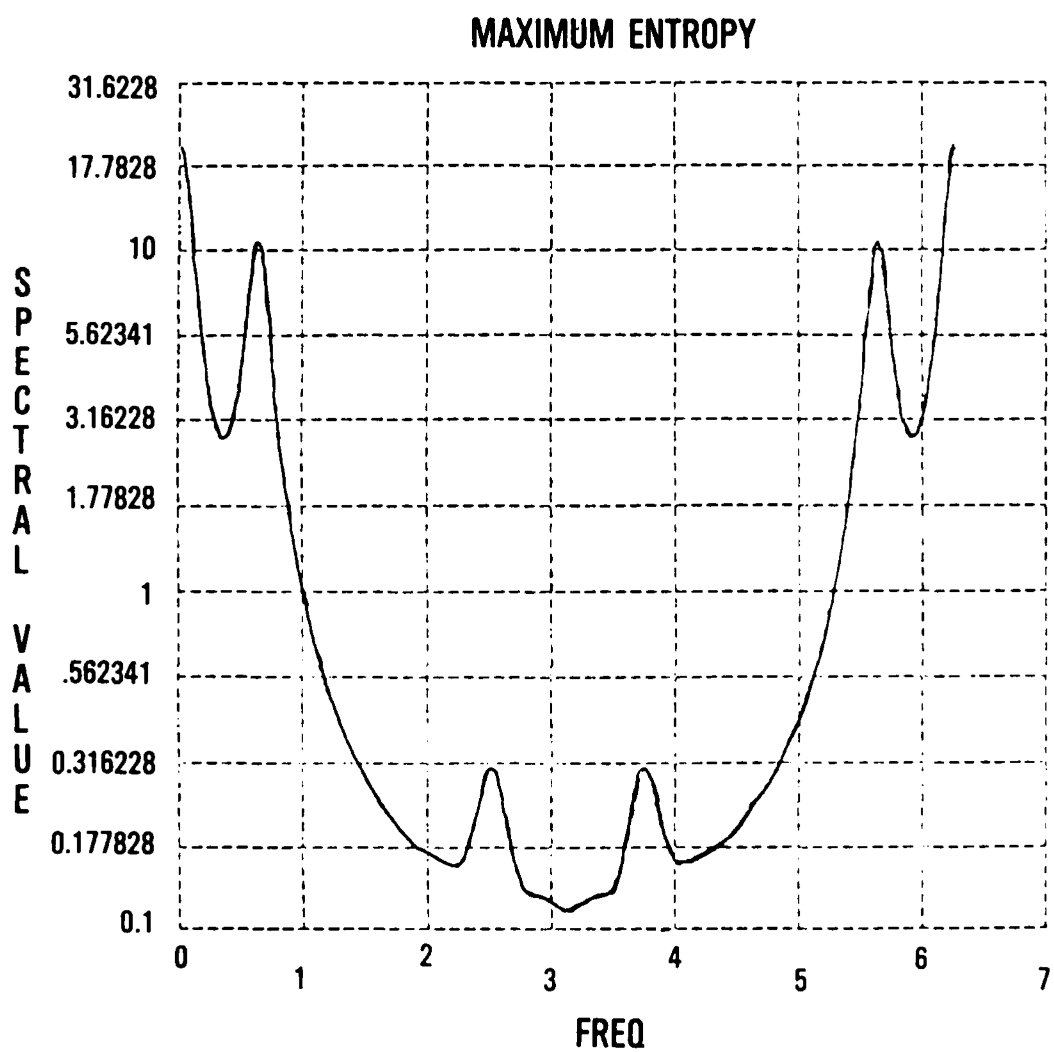

FIG. 9: SPECTRAL ESTIMATES—MAXIMUM ENTROPY METHOD

(SEE FIG. 6 FOR TEST DATA)

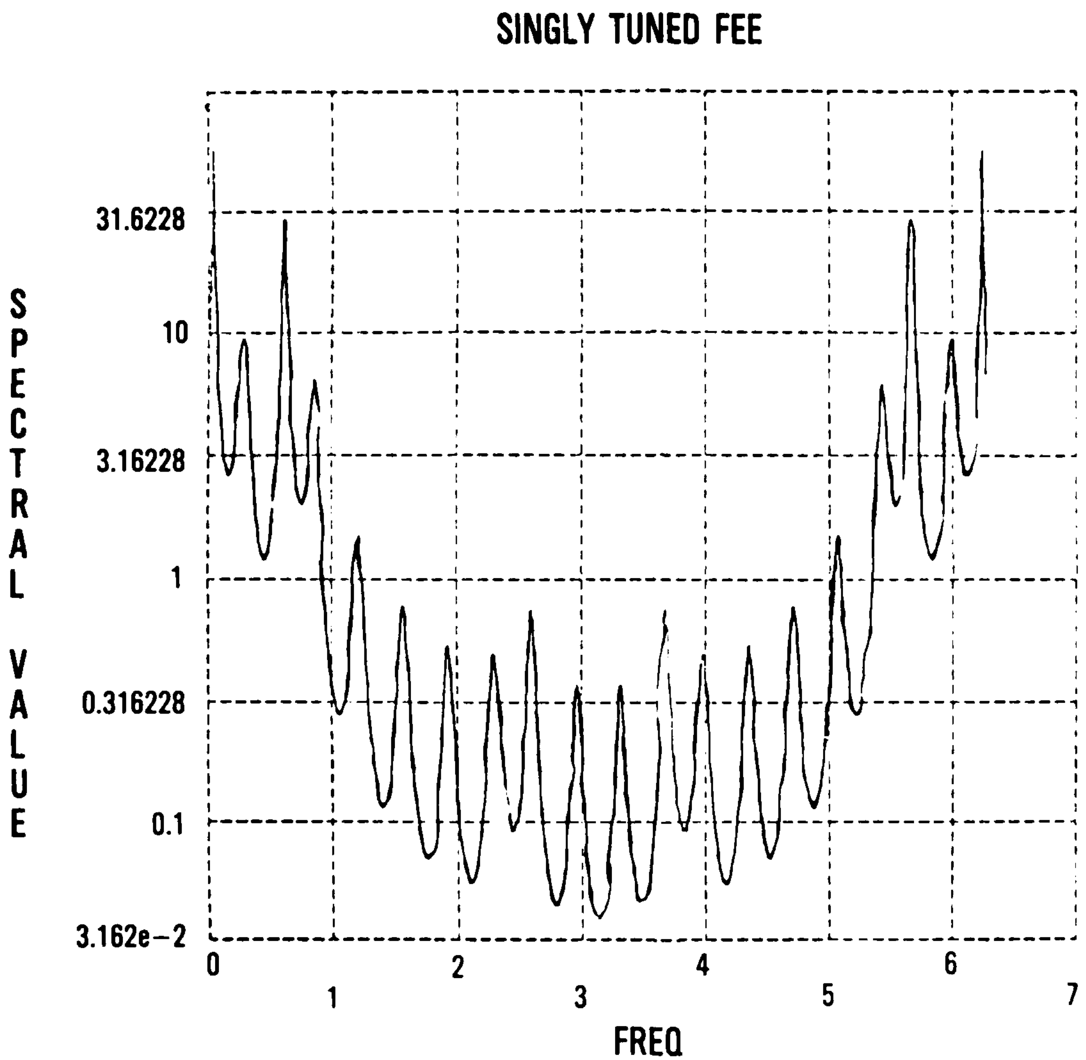

FIG. 10: SPECTRAL ESTIMATES—SINGLY TUNED FEE
(SEE FIG. 6 FOR TEST DATA)

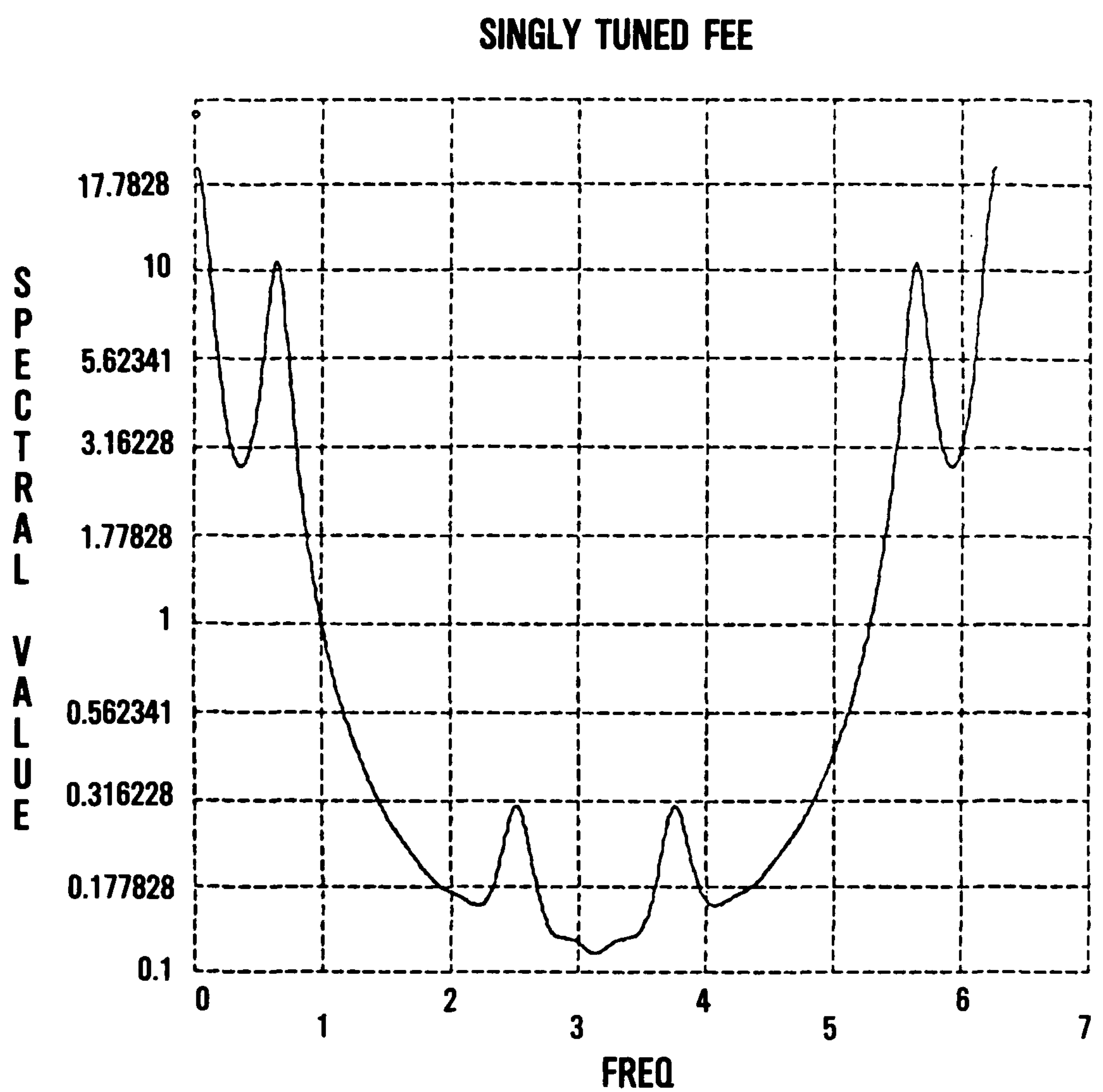

FIG. 11: SPECTRAL ESTIMATES—SINGLY TUNED FEE WITH $\mu = 0$
(NOTICE THAT THE ESTIMATES ARE SAME AS BY MAXIMUM ENTROPY (FIG. 9))

REFERENCES

1. D. C. Youla, "The FEE: A New Tunable High-Resolution Spectral Estimator. Part I - Background Theory and Derivation," Technical Note No. 3, PINY, Contract No. F30602-78-C-0048, RADC.

```
1.60p
                 dimension c(20), tn(20,20), peak(20
                 dimension coef (20), fx(1024), thi(1024)
                 character*16 header (3)
                 complex z, fz, cexp, cth
                 common/respos/ph_allpass(1024)
                 external plot_setup(descriptors)
                 external plot_(descriptors)
                 external ioa_(descriptors)
                 external read_list_(descriptors)
                 external write_list_(descriptors)
                 header (1) - "Maximum Entropy"
                 header (2) - "Singly tuned FEE"
                 header (3) - "Doubly tuned FEE"
                 call ioa_(enter two digit file number)
                 call read_list_(nfil)
                 read(nfil,113) nres
                 read(nfil,111)thmax
 113             format(I4)
                 read(nfil,110) npts
                 do 10 i=1, npts
  10             read(nfil,111) c(i)
 110             format(I2)
 111             format(E18.10)
                 nptsi=npts-1
                 do 40 jr=1,nptsi
                 do 20 i=1,jr
  20             tn(jr,i) = c(jr+1-i)
                 kd = npts-jr+1
                 do 30 i=2,kd
                 jc=jr+i-1
  30             tn(jr,jc)=c(i)
  40             continue
                 nfilot=18
                 nsign=1
                 deln=0.0
                 do 60 i=1,npts
                 call precof(nptsi,i,tn,cf)
                 coef(npts-i+1)=nsign*cf
                 dsin = (nsign*c(npts-i+1)*cf)+deln
                 nsign = -nsign
```

Appendix - Fortran Program Listing

(Appendix, continued)

```
 60        continue
           write(nfilot,114)
114        format(Sx,"Spectral Estimates follow")
           cmult = deln * coef(1)
           cmult = abs(cmult)
           cmult = 1.0 / cmult
           cmult = sqrt(cmult)
           do 75 i=1, npts
 75        coef(i) = cmult * coef(i)
           call spectral_ estimator(thi,fx,coef,thmax,amu,peak,
           npts,nres,nfes)
           do 80 i=1,nres
           write(nfilot,115) i, thi(i), fx(i), ph_allpass(i)
115        format(I4, 5x, E12.4, 4(2x,E14.5))
 80        continue
           call plot_Ssetup(header(nfee+1),"freq","spectral
           value",3,10.0,1,0)
           call plot_(thi,fx,1024,1,)
           if(nfee.eq.0)go to 95
           call ioa_("robustness factor and peak freq are")
           call write_list_(amu)
           do 90 i=1,nfee
 90        call write_list_(peak(i))
           call ioa("Input data is in file No.")
           call write_list_(nfil)
 95        continue
           call ioa_("If you wish to make a hard copy, do it now")
           call ioa_("Then type i if you wish to plot")
           call ioa_("the phase of all pass fee network")
           call ioa_("else type in 0")
           call read_list_(ntemp)
           if(ntemp .ne. 1) go to 96
           call plot_Ssetup("phase of tuned fee network","freq",
           "phase",1,10.0,1,0 /c)
           call plot_(thi,ph_allpass,1024,1," ")
 96        continue
           close(nfil)
           close(nfilot)
           stop
           end
```

(Appendix, continued)

```
          subroutine spectral_estimator(thi,fx,cf,thmax,amu
          peak,npts,nres,nfee)
          dimension thi(1024),fx(1024),peak(20),cf(20),cf-d(20)
          external ioa-(descriptors)
          external read-list-(descriptors)
          call ioa-("enter 0 for mem, 1 for singly and 2 for
          doubly tuned fee")
          call read-list-(nfee)
          if(nfee.eq.0) go to 20
          call ioa-("enter robustness factor -: amu")
          call read-list-(amu)
          do 10 i=1,nfee
          call ioa-("enter desired resonant peak(in.radians)")
          call read-list-(peak(i))
10        continue
          call design-allpass(npts,npts-d,cf,peak,cphase-d,
          cf-d)
          if(nfee.eq.1) go to 20
          if(peak(2).gt.peak(1)) go to 15
          call ioa-("error:  2nd res freq less than 1st res freq.
          Treated as singl/cy tuned")
          go to 20
15        nerror = 0
          call design-allpass-2(npts,npts d,nerror,cf,peak,
          cphase-d,cf-d)
          if(nerror .eq. 0) go to 20
          call ioa-("error:  All pass can not be designed.  Treated
          as  singly tuned/c")
20        continue
          call response (nres,nfee,npts,npts-d,cf,thmax,cf-d,
          cphase-d,amu,fx,thi)
          return
          end
          subroutine response(nres,nfee,npts,npts-d,cf,thmax,
          cf-d,cphase-d,amu,fx/c,thi)
          common/respos/ph-allpass(1024)
          nfee = 0 if mem is desired, = number of resonant
          peaks for fee)
          dimension fx(1024), thi(1024),cf(20),cf-d(20)
          delth=nres
          delth=thmax/delth
          amusq=amu*amu
          amult-num = 1.0e+00 - amusq
```

(Appendix, continued)

```
                   amult-den = 1.0e+00 +amusq
                   amu2 = amu+amu
                   do 80 i=1,nres
                 th = i*delth
                   thi(i) = th
                   cphase = 0.0e+00
 20                continue
                   a = (peak(1)-alphal)/2.0e+00
                   b = (peak(1)+alphal)/2.0e+00
                   cf-d(2) = sin(a)/sin(b)
                   cf-d(1) = 1.e+00
                   npts-d = 2
                   do 30 i=3,20
 30                cf-d(i) = 0.e+00
                   call amp-arg(npts-d,cf-d,peak(i),cphase-d,amp,phase-d,
                   all-phase-d)
                   nfil=40
101                format(2x,"Design",4x5E13.6)
                   th =0.e+00
                   close(nfil)
                   return
                   end
                   function twopi-mod(a)
                   twopi = 6.283184e+00
                   na = a/twopi
                   twopi-mod=-na*twopi +a
                   if(twopi-mod.lt.0.e+00) twopi-moe=twopi+twopi-mod
                   return
                   end
                   subroutine design-all-pass-2(npts,npts-d,nerror,
                   cf,peak,cphase-d,cf-d)
                   dimension cf(20),cf-d(20),peak(20)
              c...This subroutine assumes that singly tuned been called
                   once
                     pi = 3.141592e+00
                   twopi = 6.283184e+00
                   threepi = 9.424696e+00
                   cphase = 0.e+00
                   call amp-arg(npts,cf,peak(1),cphase,ampl,phasel,
                   all-phasel)
                   call amp-arg(npts,cf,peak(2),cphase,amp2,phase2,
                   all-phase2)
                   gammal = all-phasel + peak(1)
```

(Appendix, continued)

```
        gamma2= all-phase2 + peak(2)
        gammal = twopi-mod(gammal)
        gamma2 =twopi-mod(gamma2)
        nfil=40
        call amp-arg(npts-d,cf-d,peak(2),cphase-d,amp,
        ph2,aph2)
        call amp-arg(npts-d,cf-d,peak(1),cphase-d,al,phl,aphl)
        alphal = twopi - gammal
        alpha2 = twopi - gamma2
        eps = 0.0e + 00
        if(gammal .ge. gamma2) go to 10
        if(gammal .gt. pi) go to 100
        if(gamma2 .lt. pi) go to 100
        eps = 1.0e+00
        alphal = pi - gamma 1
        alpha2 = threepi - gamma2
10      continue
        nfil=40
        ar = -0.5e+00 * alpha2
        r = (cf-d(2)-1.e+00)*sin(peak(2)/2.0e+00)/sin(ar)
        x = (cf-d(2)+1.0e+00)*cos(peak(2)/2.0e+00)
        x = -x + (r*cos(ar))
        br = x*cos(peak(2)/2.0e+00)/(cos(peak(2))-cos(peak(1)))
        epsi = 1.e+00/(1.e+00 + br)
        epsmi = 1.0e+00 - epsi
        cf-d(3) = epsmi + (epsi*cf-d(2) )
        cf-d(2) = (epsi*(1.e+00+cf-d(2))) - 2.0e+00*cos(peak(1))
                *epsmi
        cphase-d = eps * pi
        npts-d = 3
        call amp-arg(npts-d,cf-d,peak(2),cphase-d,amp,ph2d,
        aph2d)
```

PART III

MULTI CHANNEL CASE

by

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ABSTRACT

Let $\underset{\sim}{x}_t$ denote a discrete-time, zero-mean, second-order stationary, full-rank m-dimensional random vector process. Let us assume that the spectrum of $\underset{\sim}{x}_t$ is absolutely continuous and let $K(\theta)$ and $C(k)$ denote the associated mxm spectral density and covariance matrices, respectively.

Suppose that the partial information $C(0), C(1), \ldots, C(n)$ is known in advance without error. The first major contribution of this research report is contained in theorem 2 which supplies, via Eqs. (173) and (174), a general explicit parametric formula for the class of all spectral densities $K(\theta)$ consistent with the given n+1 (matrix) pieces of covariance data.

The free parameter is an mxm matrix function $\rho(z)$ which is analytic in $|z| < 1$ and of euclidean norm less than unity almot everywhere on $|z| = 1$. Thus, $\rho(z)$ is essentially an arbitrary passive scattering matrix!

Most of the numerical calculations can be carried out efficiently and recursively. Indeed, our algorithm can be constructed around the generalized Levinson recursions for the left-right matrix orthogonal polynomials generated by the sequence $C(0), C(1), \ldots, C(n)$. Consequently, all the highly developed and sophisticated software that has come into existence during the last decade because of the intense effort devoted to Burg's maximum-entropy estimator, can be used intact.

The second major contribution is the identification of the FEE or Flat-Echo Estimator. The FEE is based squarely on frequency-domain concepts and represents an entirely new particular solution to the problem of spectrum recovery. It emerges by an appropriate restriction of the functional form of the parameter $\rho(z)$. Some of its principal advantages are the following.

1) "Numerical robustness" can be assigned in advance by the adjustment of m^2 numerical parameters.

2) It is possible to "tune" the estimator to produce selective spectral amplification at a finite number of desired frequencies without impairing either its interpolatory character or its numerical robustness. Enhanced resolution is to be expected.

3) For the setting of the numerical parameters which yields maximum robustness, namely $\rho(z) \equiv 0_m$, the FEE coincides with the MEE.

In conclusion, we should like to mention that our development of the general parametric formula for $K(\theta)$ alluded to above, constitutes the solution to a long open problem which some researchers, Burg included, have considered to be prohibitively difficult.

I. INTRODUCTION

In a previous publication [1] the author developed an entirely new solution to the problem of spectrum recovery from a given set of covariance samples of a single random process.

The flat-echo estimator (the FEE) incorporates the following important features:

1) Numerical robustness is assignable in advance by the adjustment of a scalar parameter and can be traded off against resolution.

2) It is possible to "tune" the estimator to provide selective amplification at any set of prescribed frequencies without impairing either its interpolatory character or its numerical robustness.

3) For the parameter setting which yields maximum robustness, the FEE coincides with the MEE, the maximum-entropy estimator.

Our principal objective in this continuing study is to generalize the FEE to the multichannel case and we therefore assume that the reader is familiar with the geometric and analytic points of view adopted in Ref. 1. Moreover, since this is a research report, our demands on his knowledge will vary widely but we have nonetheless attempted to be tutorial whenever possible.

With regard to the matter of notation, most of it is standard and self-explanatory. However, let us point out that $\overline{A}$, A', $A^*(\equiv \overline{A}')$ and det A denote, in the same order, the complex conjugate, transpose, adjoint and determinant of the matrix A.

Column-vectors are written $\underset{\sim}{a}$, $\underset{\sim}{x}$, etc., or as

$$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)' \qquad (0.0)$$

whenever it is desirable to exhibit the components. Further, 1_m is the mxm identity, $0_{m,k}$ the mxk zero matrix and $\underset{\sim}{0}_m$ is the m-dimensional column-vector.

As usual, if $A = A^*$ is hermitean, $A \geq 0 (> 0)$ means that $\underset{\sim}{x}^* A \underset{\sim}{x} \geq 0 (> 0)$ for every vector $\underset{\sim}{x} \neq \underset{\sim}{0}$. We then say that A is nonnegative or positive-definite, as the case may be.

In this paper, we distinguish carefully between the right-half p-plane and the interior of the unit circle since some of our arguments are couched in both analogue and digital terms.

Thus, if $W(p, z)$ denotes a matrix whose entries are meromorphic functions of both $p = \sigma + j\omega$ and $z = r e^{j\theta}$, it is convenient, for typographical reasons, to set

$$W_*(p, z) \equiv W^*(-\bar{p}, 1/\bar{z}) \quad . \tag{0.2}$$

Clearly, for $p = j\omega$ and $z = e^{j\theta}$, ω and θ real,

$$W_*(j\omega, e^{j\theta}) = W^*(j\omega, e^{j\theta}) \tag{0.4}$$

and the sub-star coincides with the top-star.

Let x stand for either p or z and let $\mathcal{R}$ denote either the open right-half p-plane, $\mathrm{Re}\, p > 0$, or the open unit-circle, $|z| < 1$.

An mxm matrix $Z(x)$ is said to be positive if 1) it is analytic in $\mathcal{R}$ and 2),

$$\frac{Z(x) + Z^*(x)}{2} \geq 0 \quad , \quad x \in \mathcal{R} \; . \tag{0.6}$$

Similarly, an mxm matrix $W(x)$ is said to be bounded if 1) it is analytic in $\mathcal{R}$ and 2),

$$1_m - W(x)W^*(x) \geq 0 \quad , \quad x \in \mathcal{R} \; . \tag{0.8}$$

If in addition, $Z(x)$ or $W(x)$ is real for all real $x \in \mathcal{R}$, it is said to be positive-real or bounded-real, respectively.

A positive matrix $Z(x)$ which is a.e. skew-hermitean on the boundary of $\mathcal{R}$ is said to be lossless while a bounded matrix $W(x)$ which is a.e. unitary on the boundary of $\mathcal{R}$ is said to be regular paraconjugate unitary. If $Z(x)$ is also real, it is Foster and if $W(x)$ is real it is regular paraunitary.

II. FORMULATION

Let $\underset{\sim}{x}_t$ denote a discrete-time m-dimensional random vector process where "t" can traverse all positive and negative integers. Let us also suppose that the process is zero-mean and second-order stationary. Then,

$$E(\underset{\sim}{x}_t \underset{\sim}{x}^*_{t+k}) = C(k) , \quad |k| = 0 \to \infty \quad , \tag{1}$$

is the associated mxm covariance function.[1] Clearly, for all integers k,

$$C(-k) = E(\underset{\sim}{x}_t \underset{\sim}{x}^*_{t-k}) = E(\underset{\sim}{x}_{t+k} \underset{\sim}{x}^*_t) = C^*(k) \quad . \tag{2}$$

As is well-known [2], if the power spectrum $F(\theta)$ of the process is absolutely continuous, there exists an mxm hermitean nonnegative-definite spectral density matrix $K(\theta)$, whose entries are absolutely integrable[2] over $-\pi \leq \theta \leq \pi$ and is such that,

$$C(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jk\theta} K(\theta) d\theta \quad . \tag{3}$$

Hence,

$$K(\theta) \sim \sum_{k=-\infty}^{\infty} C(k) e^{jk\theta} \tag{4}$$

and the mxm covariance samples $C(k)$, $|k| = 0 \to \infty$, emerge as the Fourier coefficient of $K(\theta)$.

Moreover, if the process is free of essentially determined components, $K(\theta)$ satisfies the Paley-Wiener criterion,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \det K(\theta) d\theta > -\infty \quad . \tag{5}$$

[1] Very often, we do not distinguish between the function C(k) and the particular sample C(k). Hopefully, when taken in context this should never cause confusion.

[2] For short, $K(\theta) \in L_1$.

Clearly, such a process automatically obeys the full-rank condition,

$$\det K(\theta) \neq 0 \quad , \quad \text{a.e.,} \tag{5a}$$

but not conversely.

From the inequality $K(\theta) \geq 0$ it is easily shown [6] that

$$T_n \equiv \begin{bmatrix} C(0) & C(1) & \ldots C(n) \\ C(-1) & C(0) & \ldots C(n-1) \\ \cdots & \cdots & \cdots \\ C(-n) & C(-n+1) & \ldots C(0) \end{bmatrix} \geq 0, \quad n = 0 \rightarrow \infty \ . \tag{5b}$$

Furthermore, if the process is full-rank (and, a fortiori, if it satisfies Paley-Wiener), only the inequality sign prevails.

Unless stated explicitly otherwise, we shall adhere to two main working assumptions.

A_1) The power spectrum $F(\theta)$ of the process is absolutely continuous, i.e.,

$$dF(\theta) = K(\theta)d\theta \tag{6}$$

where $K(\theta) \epsilon L_1$ and

$$K(\theta) = K^*(\theta) \geq 0 \ . \tag{7}$$

A_2) Either the process is full-rank or its satisfies the Paley-Wiener condition (5) or both. The precise situation can always be ascertained from the context.

Our primary task is to develop a physically insightful solution to the problem of estimating $K(\theta)$ from a knowledge of n+1 given error-free mxm covariance samples C(0), C(1), ..., C(n).

III. THE 2m-PORT INTERPOLATORY CASCADE

By way of motivation, let us recall the properties of a cascade N of n ideal TEM lines which have the same 1-way delay $\tau > 0$ but arbitrary positive characteristic impedances $R_0, R_1, \ldots, R_{n-1}$ [1].

Let $S_N(p)$ denote the scattering matrix of N normalized to R_0 at the input and any positive port number at the output. Then, $S_N(p)$ is completely parametrized by the specification of a single arbitrary real polynomial h(z) of the form[3]

$$h(z) = zh_1 + z^2h_2 + \ldots + z^n h_n . \tag{8}$$

Explicitly,

C_1) with h(z) so prescribed, choose the real polynomial g(z) (of degree $\leq n-1$) to be the strict-Hurwitz solution of the equation

$$g(z)g_*(z) = 1 + h(z)h_*(z) \quad . \tag{9}$$

This Wiener-Hopf factor is analytic together with its inverse in $|z| \leq 1$ and is uniquely determined under the restriction $g(0) > 0$.

C_2) Let

$$z = e^{-2p\tau} . \tag{9a}$$

Then,

$$S_N(p) = \frac{1}{g(z)} \left[\begin{array}{c|c} h(z) & e^{-np\tau} \\ \hline e^{-np\tau} & -z^n h_*(z) \end{array} \right] . \tag{10}$$

It is readily seen that $S_N(p)$ is a real meromorphic matrix function of p that is analytic in $\mathrm{Re}\, p \geq 0$ and satisfies the identity

$$S_N(p)S_{N*}(p) = 1_2 \quad . \tag{11}$$

Thus, $S_N(p)$ is regular paraunitary.

---

[3] The property $h(0) = 0$ reflects the fact that the input-side normalization number has been chosen equal to R_0, the characteristic impedance of the first line.

The immediate goal is to generalize the above construction to the 2m-port case, $m \geq 1$. However, due to the lack of commutativity of matrix multiplication, several subtleties arise for $m > 1$ which can only be dealt with by modifying the procedure in a substantial way.

Lemma 1. Let $H(z)$ denote an arbitrary (and not necessarily real) mxm polynomial matrix of degree $\leq n$ of the form

$$H(z) = zH_1 + z^2 H_2 + \cdots + z^n H_n \quad . \tag{12}$$

There exist two mxm polynomial matrices $G_1(z)$ and $G_2(z)$ of degrees $\leq n-1$ such that

1) $\det G_1(z) \cdot \det G_2(z) \neq 0, \quad |z| \leq 1 \; .$ (13)

2) The matrix

$$L(z) = G_1^{-1}(z)H(z)G_2(z) \tag{14}$$

is polynomial.

3) Under the identification $z = e^{-2p\tau}$, $\tau > 0$, the 2mx2m matrix

$$S_N(p) = \left[\begin{array}{c|c} G_1^{-1}(z)H(z) & e^{-np\tau}G_1^{-1}(z) \\ \hline e^{-np\tau}G_2^{-1}(z) & -z^n G_2^{-1}(z)H_*(z) \end{array}\right] \tag{15}$$

is meromorphic in p, analytic in $\mathrm{Re}\, p \geq 0$ and paraconjugate unitary:

$$S_N(p)S_{N*}(p) = 1_{2m} \quad . \tag{16}$$

Proof. Let $G_1(z)$ and $G_2(z)$ be chosen as the mxm Wiener-Hopf factors of the two matrix equations

$$G_1(z)G_{1*}(z) = 1_m + H(z)H_*(z) \quad , \tag{17}$$

$$G_2(z)G_{2*}(z) = 1_m + H_*(z)H(z) \quad . \tag{18}$$

Clearly, because of the form (12) of $H(z)$, both $H(z)H_*(z)$ and $H_*(z)H(z)$ are paraconjugate hermitean matrices with powers of z restricted to lie between $-(n-1)$ and $n-1$.

From a classical result [3], $G_1(z)$ and $G_2(z)$ are polynomial and uniquely determined up to multiplication on the right by arbitrary constant mxm unitary matrices. In addition, their degrees do not exceed n-1 and they are minimum-phase, i.e.,

$$\det G_1(z) \cdot \det G_2(z) \neq 0 \ , \quad |z| < 1 \quad . \tag{19}$$

But, as it is obvious that the right-hand sides of (17) and (18) are also nonsingular for all $z = e^{j\theta}$, θ real, we actually have the stronger inequality (13) which <u>includes</u> the boundary of the unit-circle, $|z| = 1$.

If we now observe that

$$S_N(p) = \left[\begin{array}{c|c} G_1^{-1}(z) & 0_m \\ \hline 0_m & G_2^{-1}(z) \end{array}\right] \cdot \left[\begin{array}{c|c} H(z) & e^{-np\tau}1_m \\ \hline e^{-np\tau}1_m & -z^n H_*(z) \end{array}\right] \ , \tag{20}$$

it is straightforward to verify (16) with the aid of (17) and (18). Thus, $S_N(p)$ is meromorphic and regular paraconjugate unitary.[4]

Since a right-inverse is also a left-inverse, (16) implies that

$$S_{N*}(p)S_N(p) = 1_{2m} \quad . \tag{21}$$

In turn, (21) yields[5]

$$(G_2G_{2*})^{-1} + H_*(G_1G_{1*})^{-1}H = 1_m \ , \tag{22}$$

$$H(G_2G_{2*}) - (G_1G_{1*})H = 0_m \ , \tag{23}$$

$$(G_1G_{1*})^{-1} + H(G_2G_{2*})^{-1}H_* = 1_m \quad . \tag{24}$$

From (23),

$$G_1^{-1}(z)H(z)G_2(z) \equiv L(z) = G_{1*}(z)H(z)G_{2*}^{-1}(z) \quad . \tag{25}$$

---

[4]It follows from (13) that $G_1^{-1}(z)$ and $G_2^{-1}(z)$ are analytic in $|z| \leq 1$ so that the domain of analyticity of $S_N(p)$ <u>includes</u> the $j\omega$-axis in the p-plane.

[5]Arguments are omitted wherever convenient.

But the right-hand side of (25) is analytic in $|z| \geq 1$ whereas the left-hand side is analytic in $|z| \leq 1$ and this means that L(z) is analytic in the entire finite z-plane. By Liouville's theorem, L(z) is a polynomial matrix and the proof of lemma 1 is complete, Q.E.D.

Comment 1. If H(z) is chosen real, $G_1(z)$ and $G_2(z)$ can also be constructed real and $S_N(p)$ is a real regular paraunitary matrix.

Let us now suppose that a 2m-port N of generic type defined in lemma 1 is closed on its output side on a passive m-port load with normalized scattering matrix description $\Gamma(z)$. Of course, $\Gamma(z)$ is mxm and bounded and therefore analytic in $|z| < 1$.

Let the four mxm partitions in (15) be labeled $S_{11}(p)$, $S_{12}(p)$, $S_{21}(p)$ and $S_{22}(p)$. Then, from first principles [4],

$$S = S_{11} + S_{12}\Gamma(1_m - S_{22}\Gamma)^{-1}S_{21} \tag{26}$$

is the normalized scattering matrix of the resultant m-port N_m.

Thus, making the appropriate substitutions and using (17),

$$S(z) = G_1^{-1}H + z^n G_1^{-1}\Gamma(1_m + z^n G_2^{-1}H_* \Gamma)^{-1}G_2^{-1} \tag{27}$$

$$= G_1^{-1}(HG_2(1_m + z^n G_2^{-1}H_* \Gamma) + z^n \Gamma)(G_2 + z^n H_* \Gamma)^{-1} \tag{28}$$

$$= G_1^{-1}(HG_2 + z^n(1_m + HH_*)\Gamma)(G_2 + z^n H_* \Gamma)^{-1} \tag{29}$$

$$= G_1^{-1}(HG_2 + z^n G_1 G_{1*}\Gamma)(G_2 + z^n H_* \Gamma)^{-1} \tag{30}$$

$$= (L + z^n G_{1*}\Gamma)(G_2 + z^n H_* \Gamma)^{-1} \tag{31}$$

$$= (L(z) + \tilde{G}_1(z)\Gamma(z))(G_2(z) + \tilde{H}(z)\Gamma(z))^{-1} \tag{32}$$

where $L(z) = G_1^{-1}(z)H(z)G_2(z)$ and[6]

$$\tilde{G}_1(z) \equiv z^n G_{1*}(z) \quad , \tag{33}$$

---

[6] $\tilde{G}_1(z)$ and $\tilde{H}(z)$ are the mxm matrix polynomials reciprocal to $G_1(z)$ and H(z), respectively.

$$\tilde{H}(z) \equiv z^n H_*(z) \quad . \tag{34}$$

Consequently, since it has been shown in lemma 1 that L(z) is polynomial,

$$S = (L+\tilde{G}_1 \Gamma)(G_2+\tilde{H}\Gamma)^{-1} \tag{35}$$

appears as a bilinear matrix transform of Γ with polynomial matrix coefficients that depend on z alone.

Lemma 2. Let

$$\Gamma(z) = z\rho(z) \tag{36}$$

where $\rho(z)$ is an arbitrary mxm bounded matrix in z.[7] Then, the first n+1 coefficients in the power-series expansion of

$$S = (L+z\tilde{G}_1 \rho)(G_2+z\tilde{H}\rho)^{-1} \tag{37}$$

about $z = 0$ are independent of the choice of $\rho(z)$.

Proof. Let $S_o(z)$ denote S(z) with $\rho(z) \equiv 0_m$. It suffices to prove that

$$S(z) - S_o(z) = 0(z^{n+1}) \tag{38}$$

in the neighborhood of $z = 0$.

Evidently,

$$S_o(z) = L(z)G_2^{-1}(z) = G_1^{-1}(z)H(z) \tag{39}$$

since $G_1 L = HG_2$. Thus,

$$S-S_o = (L+z\tilde{G}_1\rho)(G_2+z\tilde{H}\rho)^{-1}-G_1^{-1}H \tag{40}$$

so that

$$G_1(S-S_o)(G_2+z\tilde{H}\rho) = G_1(L+z\tilde{G}_1\rho) - H(G_2+z\tilde{H}\rho) \tag{41}$$

$$= z(G_1\tilde{G}_1 - H\tilde{H})\rho \tag{42}$$

$$= z^{n+1}\rho \quad , \tag{43}$$

[7]Such a $\Gamma(z)$ is automatically bounded. Moreover, every S(z) generated by (37) is also bounded if $\rho(z)$ is bounded.

from (17). Or,

$$S - S_o = z^{n+1} G_1^{-1} \rho (G_2 + z\tilde{H}\rho)^{-1} \quad . \tag{44}$$

However, because $G_1(0)$ and $G_2(0)$ are nonsingular and $\rho(z)$ is analytic in $|z| \leq 1$, (44) implies the desired result (38), Q.E.D.

Comment 2. Since S(z) is bounded,

$$Z(z) = (1_m + S(z))(1_m - S(z))^{-1} \tag{45}$$

is positive whenever it exists. Clearly, as is seen from (37) and (12),

$$S(0) = L(0)G_2^{-1}(0) = G_1^{-1}(0)H(0) = 0_m \tag{46}$$

so that neither $1_m + S(z)$ nor $1_m - S(z)$ are identically singular. Thus, Z(z) and $Z^{-1}(z)$ are both well-defined.

Let us write

$$Z(z) = 1_m + \sum_{k=1}^{n} 2C(k)z^k + 0(z^{n+1}) \quad . \tag{47}$$

According to lemma 2, the mxm coefficients C(k), $k = 1 \rightarrow n$, are independent of the particular choice of bounded $\rho(z)$ and are therefore calculable from the power-series expansion of

$$Z_o(z) = (1_m + S_o(z))(1_m - S_o(z))^{-1} \, . \tag{48}$$

Since the factors in (48) commute we can also write

$$Z_o = (1_m - S_o)^{-1}(1_m + S_o) \tag{49}$$

$$= (1_m - G_1^{-1}H)^{-1}(1_m + G_1^{-1}H) \tag{50}$$

$$= (G_1 - H)^{-1}(G_1 + H) = 2P_n^{-1}Q_n \tag{51}$$

where

$$P_n(z) = G_1(z) - H(z) \tag{52}$$

and

$$Q_n(z) = \frac{G_1(z) + H(z)}{2} \quad . \tag{53}$$

By direct calculation and Eq. (17),

$$P_n(z)Q_{n*}(z) + Q_n(z)P_{n*}(z) = 1_m \tag{54}$$

and we are now ready for the remarkable lemma 3.

Lemma 3. 1) The polynomial matrices $P_n(z)$ and $Q_n(z)$ are nonsingular in the closed unit circle, i.e.,

$$\det P_n(z) \cdot \det Q_n(z) \neq 0 \quad , \quad |z| \leq 1 \quad . \tag{55}$$

$$2) \quad \frac{Z_o(z) + Z_{o*}(z)}{2} = (P_{n*}(z)P_n(z))^{-1} \quad . \tag{56}$$

3) The coefficients $C(k)$, $k = 1 \rightarrow n$, determine the interpolatory 2m-port N completely up to within a frequency insensitive all-pass cascaded at the output side.

Proof. 1) Suppose first that for some z_o, $|z_o| < 1$, $\det P_n(z_o) = 0$. Then, there exists a nontrivial m-vector $\underset{\sim}{a}$ such that $\underset{\sim}{a}^* P_n(z_o) = \underset{\sim}{0}'_m$.

Thus, from (51) and the analyticity of $Z_o(z)$ in $|z| < 1$,

$$\underset{\sim}{a}^* P_n(z_o) Z(z_o) = \underset{\sim}{0}'_m = 2\underset{\sim}{a}^* Q_n(z_o) \quad . \tag{57}$$

Therefore, invoking (52) and (53),

$$\underset{\sim}{a}^*(P_n(z_o) + 2Q_n(z_o)) = \underset{\sim}{a}^* G_1(z_o) = \underset{\sim}{0}'_m \tag{58}$$

which is impossible because $G_1(z_o)$ is nonsingular in $|z| \leq 1$.

On the other hand, if $|z_o| = 1$, $P_{n*}(z_o) = P_n^*(z_o)$, $Q_{n*}(z_o) = Q_n^*(z_o)$ and (54) evaluated at $z = z_o$ reads

$$P_n(z_o)Q_n^*(z_o) + Q_n(z_o)P_n^*(z_o) = 1_m \quad . \tag{59}$$

Thus, by multiplying (59) on the left and right by $\underset{\sim}{a}^*$ and $\underset{\sim}{a}$, respectively, we obtain $\underset{\sim}{a}^*\underset{\sim}{a} = 0$, another contradiction.

2) According to (51) and (54),

$$\frac{Z_o + Z_{o*}}{2} = P_n^{-1}(P_n Q_{n*} + Q_n P_{n*}) P_{n*}^{-1} = (P_{n*} P_n)^{-1} \quad . \tag{60}$$

3) Let

$$P_n(z) = A_0 + A_1 z + \cdots + A_n z^n \quad , \tag{61}$$

where the A's remain to be determined in terms of the C's.

From (60),

$$\frac{Z_o(z) + Z_{o*}(z)}{2} \cdot \tilde{P}_n(z) = z^n \cdot P_n^{-1}(z) \tag{62}$$

where

$$\tilde{P}_n(z) = z^n P_{n*}(z) = A_n^* + A_{n-1}^* z + \cdots + A_o^* z^n \quad . \tag{63}$$

Part 1) permits us to conclude that $Z_o(z)$ and $P_n^{-1}(z)$ are both analytic in $|z| \leq 1$ and it is therefore legitimate to set $z = e^{j\theta}$, θ real, and to equate coefficients of $e^{jk\theta}$ on both sides of (62), all k.

Carrying this out for $k = 0 \rightarrow n$ we obtain the linear system,

$$A_0^* X T_n = [1_m \; 0_m \cdots 0_m] \tag{64}$$

where

$$X = [A_0 \; A_1 \cdots A_n] \tag{65}$$

and[8]

$$T_n = \left[\begin{array}{c|cc} 1_m & C(1) & \ldots C(n) \\ \hline C(-1) & 1_m & \ldots C(n-1) \\ \cdots\cdot & \cdots\cdots\cdots & \cdots\cdots \\ C(-n) & C(-n+1) & \ldots 1_m \end{array}\right] \equiv \left[\begin{array}{c|c} 1_m & E_n \\ \hline E_n^* & T_{n-1} \end{array}\right] . \tag{66}$$

---

[8] For any k, $C(-k) \equiv C^*(k)$. Matrices which possess the structure in (66) are said to be normalized block Toeplitz.

Since

$$K_o(\theta) \equiv \frac{Z_o(e^{j\theta}) + Z_o^*(e^{j\theta})}{2} = (P_n^*(e^{j\theta})P_n(e^{j\theta}))^{-1} \tag{67}$$

is obviously positive-definite and L_1 over $-\pi \leq \theta \leq \pi$, and since it also satisfies the Paley-Wiener criterion (5), it follows [6] that

$$T_n = T_n^* > 0 \quad . \tag{68}$$

The solution of (64) is now routine.

Let A_0 be determined from the Gauss factorization

$$A_0^* A_0 = (1_m - E_n T_{n-1}^{-1} E_n^*)^{-1} \tag{69}$$

and let $(T_n^{-1})_{1;}$ denote the matrix formed with the first m rows of T_n^{-1}. Then,

$$X = (A_0^*)^{-1} \cdot (T_n^{-1})_{1;} \quad . \tag{70}$$

Equation (69) fixes A_0 uniquely up to multiplication on the left by a constant mxm unitary matrix U. In view of (70), $P_n(z)$ is then also unique up to within the same pre-factor. The next step is to produce a formula for the coefficients of $Q_n(z)$.

Let

$$Q_n(z) = F_0 + F_1 z \cdots + F_n z^n \tag{70a}$$

and let

$$Y = [F_0\ F_1 \cdots F_n] \ . \tag{71}$$

From (51),

$$2Q_n(z) = P_n(z) Z_o(z) \tag{72}$$

and a comparison of coefficients of like power of z^k, $k = 0 \rightarrow n$, yields

$$2Y = X\Sigma_n \tag{73}$$

where

$$
\Sigma_n = \begin{bmatrix} 1_m & 2C(1) & \cdots & 2C(n) \\ & 1_m & & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \\ & & & \cdot\ 2C(1) \\ & & & 1_m \end{bmatrix} . \tag{74}
$$

Thus, $Q_n(z)$ is also uniquely determined up to within the unitary premultiplier U.

Consequently, $H(z) = \frac{1}{2} P_n(z) - Q_n(z)$ and $G_1(z) = \frac{1}{2} P_n(z) + Q_n(z)$ are both determined up to multiplication on the left by U, and then $G_2(z)$ can be found as the minimum-phase polynomial solution of (18). As mentioned previously, $G_2(z)$ is unique up to multiplication on the right by an arbitrary constant mxm unitary matrix V.

If we now return to (15), it is immediately seen that $S_{11}(p)$ is unique, $S_{12}(p)$ is unique up to the right multiplier U^*, $S_{21}(p)$ up to the left multiplier V^* and $S_{22}(p)$ up to the left-right multipliers V^* and U^*, respectively. But this can be interpreted to mean [4] that every compatible interpolatory 2m-port N is obtained from a particular one, N_p, say, by cascading N_p on the right in a 2m-port all-pass with normalized scattering description

$$
\left[\begin{array}{c|c} 1_m & U^* \\ \hline V^* & 1_m \end{array}\right] . \tag{75}
$$

This completes the proof, Q.E.D.

The preceding lemma suggests a very natural definition.

Def. 1. A set of mxm matrices C(k), $k = 1 \to n$, is said to admit the <u>strong interpolation property</u> (the SIP), if there exist two mxm polynomial matrices $P_n(z)$ and $Q_n(z)$ of degrees less or equal to n such that,

I_1. $\det P_n(z) \cdot \det Q_n(z) \neq 0, \quad |z| \leq 1$;

I_2. $P_n(z) Q_{n*}(z) + Q_n(z) P_{n*}(z) = 1_m$;

I_3. For $|z| \leq 1$,

$$Z_o(z) \equiv 2P_n^{-1}(z)Q_n(z) = 1_m + \sum_{k=1}^{n} 2C(k)z^k + 0(z^{n+1}) \quad ;$$

I_4. $Z_o(z)$ is positive.

Lemma 4. A set of mxm matrices $C(k)$, $k = 1 \rightarrow n$, admits the SIP iff

$$T_n = \begin{bmatrix} 1_m & C(1) & \cdots & C(n) \\ C(-1) & 1_m & \cdots & C(n-1) \\ \cdots & \cdots & \cdots & \cdots \\ C(-n) & C(-n+1) & \cdots & 1_m \end{bmatrix} > 0 \ . \tag{76}$$

Proof. Let the data admit the SIP. Then, from I_1, I_2, and the definition of $Z_o(z)$,

$$K_o(\theta) \equiv \frac{Z_o(e^{j\theta}) + Z_o^*(e^{j\theta})}{2} = (P_n^*(e^{j\theta})P_n(e^{j\theta}))^{-1} \tag{77}$$

is L_1 and positive-definite over $-\pi \leq \theta \leq \pi$.

Furthermore, in view of I_3,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\theta} K_o(\theta) d\theta = C(k) \quad , \quad -n \leq k \leq n \quad , \tag{78}$$

and since it is obvious that $K_o(\theta)$ is nonsingular for all θ, it follows from the discussion at the end of Section II that $T_n > 0$.

Sufficiency. A review of the proof of lemma 3 shows that the polynomials $P_n(z)$ and $Q_n(z)$ must be given by the formulas,

$$P_n(z) = X\xi(z) \ , \quad Q_n(z) = Y\xi(z) \tag{79}$$

where X and Y are determined by means of Eqs. (69)-(74) and

$$\xi(z) \equiv [1_m \ z1_m \ . \ . \ . \ z^n 1_m]' \quad . \tag{80}$$

Of course, the positive-definiteness of T_n guarantees that $P_n(z)$ and $Q_n(z)$ are well-defined and it remains to be shown that they possess all the requisite properties.

Note first that I_2 is equivalent to

$$\frac{1}{2} X(\xi \xi_* \Sigma_n^* + \Sigma_n \xi \xi_*) X^* = 1_m \quad . \tag{81}$$

But, as is easily seen,

$$\xi(z)\xi_*(z) = 1 + \Omega(z) + \Omega_*(z) \tag{82}$$

where

$$\Omega(z) = \begin{bmatrix} 0_m & & & & \\ z1_m & 0_m & & & \\ z^2 1_m & z1_m & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ z^n 1_m & z^{n-1} 1_m & \ldots & z1_m & 0_m \end{bmatrix} . \tag{83}$$

Hence, since

$$T_n = \frac{1}{2}(\Sigma_n + \Sigma_n^*) \tag{84}$$

and Ω_* commutes with Σ_n,[9] we find that

$$\frac{\xi\xi_*\Sigma_n^* + \Sigma_n \xi\xi_*}{2} = T_n + \frac{1}{2}(\Omega + \Omega_*)\Sigma_n^* + \frac{1}{2}\Sigma_n(\Omega + \Omega_*) \tag{85}$$

$$= T_n + \Omega_* \cdot \frac{\Sigma_n + \Sigma_n^*}{2} + \frac{\Sigma_n + \Sigma_n^*}{2} \cdot \Omega \tag{86}$$

$$= T_n + \Omega_* T_n + T_n \Omega \quad . \tag{87}$$

[9] To a great extent, it is this commutativity that is responsible for so many of the remarkable properties exhibited by Toeplitz matrices.

Now, by using the explicit formula $X = (A_0^*)^{-1} \cdot (T_n^{-1})_{1;}$ we obtain, in succession,

$$XT_n X^* = (A_0^*)^{-1} \cdot (A_0^*) = 1_m \quad , \tag{88}$$

$$XT_n \Omega = (A_0^*)^{-1} \cdot [1_m \; 0_m \; . \; . \; . \; 0_m] \Omega = 0 \tag{89}$$

and

$$X(T_n + \Omega_* T_n + T_n \Omega) X^* = 1_m \quad . \tag{90}$$

Thus, I_2 is validated.

The remaining three properties can be disposed of almost simultaneously by proving that $Z_o(z) = 2P_n^{-1}(z)Q_n(z)$ is positive in $|z| < 1$.

First, $P_n(0) = A_0$. But, in view of (69) and the premise $T_n > 0$, A_0 is nonsingular so that $\det P_n(z) \not\equiv 0$. Thus, $Z_o(z)$ is well-defined and analytic in some neighborhood of $z = 0$.

By construction, its power series expansion about $z = 0$ must be of the form

$$Z_o(z) = 1_m + \sum_{k=1}^{\infty} 2C(k) z^k \tag{91}$$

where $C(1), C(2), \ldots, C(n)$ constitute the prescribed data. Let $r > 0$ denote the radius of convergence of (91). Of course, since we have not yet established that $P_n(z)$ is minimum-phase, i.e., that $\det P_n(z) \neq 0$, $|z| < 1$, it cannot be asserted that $r \geq 1$.

Nevertheless, by equating coefficients on both sides of the identity

$$\left(\sum_{\ell=0}^{n} A_\ell z^\ell\right) \cdot \left(1_m + \sum_{k=1}^{\infty} 2C(k) z^k\right) = 2 \sum_{\ell=0}^{n} F_\ell z^\ell \tag{92}$$

we obtain the formulas,

$$\Sigma_\ell = 2\alpha_\ell^{-1}\beta_\ell \;, \quad \ell = 0 \to \infty \quad , \tag{93}$$

where Σ_ℓ is given by (74) with n replaced by ℓ,

$$\alpha_\ell = \begin{bmatrix} A_0 & A_1 & \cdots & A_\ell \\ & A_0 & \cdots & A_{\ell-1} \\ & & \ddots & \vdots \\ & & & A_0 \end{bmatrix} \tag{94}$$

and

$$\beta_\ell = \begin{bmatrix} F_0 & F_1 & \cdots & F_\ell \\ & F_0 & \cdots & F_{\ell-1} \\ & & \ddots & \vdots \\ & & & F_0 \end{bmatrix} \quad . \tag{95}$$

(The matrices α_ℓ and β_ℓ are upper-triangular block Toeplitz.)

Our aim is to prove that

$$T_\ell \equiv \tfrac{1}{2}(\Sigma_\ell + \Sigma_\ell^*) = \alpha_\ell^{-1}(\alpha_\ell\,\beta_\ell^* + \beta_\ell\,\alpha_\ell^*)(\alpha_\ell^*)^{-1} > 0 \tag{96}$$

for all $\ell \geq 0$. Or, equivalently, that

$$\gamma_\ell \equiv \alpha_\ell\,\beta_\ell^* + \beta_\ell\,\alpha_\ell^* > 0 \quad , \quad \ell = 0 \to \infty \quad . \tag{97}$$

By hypothesis, $T_n > 0$ and (97) is correct for $\ell = 0 \to n$. Clearly then, the desired result $\gamma_\ell > 0$, $\ell \geq n$, follows trivially if it can be shown that for all such ℓ,

$$\gamma_\ell = 1_{m(\ell-n+1)} \dotplus \gamma_{n-1} \quad . \tag{98}$$

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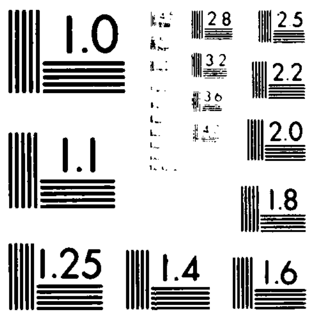

MICROCOPY RESOLUTION TEST CHART

To accomplish this we shall exploit the already-derived "even-part" condition,

$$P_n(z)Q_{n*}(z) + Q_n(z)P_{n*}(z) = 1_m \quad . \tag{99}$$

We leave it to the reader to show that a direct comparison of like powers of z on both sides of (99) leads to the 1-step update,

$$\gamma_n = 1_m \dotplus \gamma_{n-1} \,. \tag{100}$$

Since (100) is (98) with $\ell = n$, the natural way to proceed is by induction.

Evidently,

$$\alpha_{\ell+1} = \left[\begin{array}{c|c} A_0 & A_{12} \\ \hline & \alpha_\ell \end{array}\right] \quad \text{and} \quad \beta_{\ell+1} = \left[\begin{array}{c|c} F_0 & F_{12} \\ \hline & \beta_\ell \end{array}\right] \tag{101}$$

where[10]

$$A_{12} = [A_1 A_2 \cdots A_{\ell+1}] \tag{102}$$

and

$$F_{12} = [F_0 F_1 \cdots F_{\ell+1}] \quad . \tag{103}$$

By direct calculation,

$$\gamma_{\ell+1} = \alpha_{\ell+1}\beta^*_{\ell+1} + \beta_{\ell+1}\alpha^*_{\ell+1} = \left[\begin{array}{c|c} \gamma_{11} & \gamma_{12} \\ \hline \gamma^*_{12} & \gamma_\ell \end{array}\right] \tag{104}$$

in which,

$$\gamma_{11} = \sum_{k=0}^{\ell+1} (A_k F^*_k + F_k A^*_k) \tag{105}$$

and

$$\gamma_{12} = [A_1 A_2 \cdots A_{\ell+1}]\beta^*_\ell + [F_1 F_2 \cdots F_{\ell+1}]\alpha^*_\ell . \tag{106}$$

---

[10] For $\ell > n$, $A_\ell = F_\ell \equiv 0_m$.

If we now take stock of the previous footnote, it is quickly observed that (105) and (106) reduce to

$$\gamma_{11} = \sum_{k=0}^{n} (A_k F_k^* + F_k A_k^*) \tag{107}$$

and

$$\gamma_{12} = [A_1 A_2 \cdots A_n]\, \beta_{n-1}^* + [F_1 F_2 \cdots F_n]\, \alpha_{n-1}^* \quad , \tag{108}$$

respectively.

However, by equating the (1, 1) and (1, 2) blocks on the two sides of (100) it is found that $\gamma_{11} = 1_m$ and $\gamma_{12} = 0$ whence,

$$\gamma_{\ell+1} = 1_m \dotplus \gamma_{\ell} \quad . \tag{109}$$

But from the induction hypothesis,

$$\gamma_{\ell} = 1_{m(\ell-n+1)} \dotplus \gamma_{n-1} \tag{110}$$

so that (109) expands into

$$\gamma_{\ell+1} = 1_m \dotplus 1_{m(\ell-n+1)} \dotplus \gamma_{n-1} = 1_{m(\ell-n)} \dotplus \gamma_{n-1} \tag{111}$$

and the induction step is completed. Thus,

$$T_{\ell} > 0 \ , \quad \ell = 0 \rightarrow \infty \quad . \tag{112}$$

According to a generalization of a theorem of Schur (Appendix A), (112) is a sufficient condition for the power-series

$$1_m + \sum_{k=1}^{\infty} 2C(k) z^k \tag{113}$$

to define a positive function of z in $|z| < 1$.

However, since this series and $Z_o(z)$ coincide in $|z| < r$, they must, by analytic continuation, coincide in all of $|z| < 1$. We have therefore shown that $Z_o(p) = 2P_n^{-1}(z)Q_n(z)$ is positive and it remains to check property I_1.

The positivity of $Z_o(z)$ automatically guarantees its analyticity in $|z| < 1$. Hence, as shown in lemma 3, if for some z_o, $|z_o| < 1$, there exists a nontrivial vector $\mathbf{a}$ such that $\mathbf{a}^* P_n(z_o) = \mathbf{0}'_m$, then necessarily, $\mathbf{a}^* Q_n(z_o) = \mathbf{0}'_m$.

Consequently, from (99),

$$\mathbf{a}^* P_n(z_o)\tilde{Q}_n(z_o) + \mathbf{a}^* Q_n(z_o)\tilde{P}_n(z_o) = z_o^n \mathbf{a}^* = \mathbf{0}'_m \tag{114}$$

which implies that $z_o = 0$. But this is a contradiction because $P_n(0) = A_o$ is nonsingular.

Similarly, since it follows immediately from Eq. (59) that $P_n(z)$ and $Q_n(z)$ are both nonsingular on the boundary of the unit circle, the assumption $|z_o| = 1$ is equally untenable. Finally, by observing that

$$Z_o^{-1}(z) = \frac{1}{2} Q_n^{-1}(z) P_n(z) \tag{115}$$

is also positive, we can apply exactly the same reasoning to establish the nonsingularity of $Q_n(z)$ in $|z| \leq 1$, Q.E.D.

Corollary. Let the mxm matrices $C(k)$, $k = 1 \rightarrow n$, admit the SIP. Then, there exists an interpolatory 2m-port N such that

$$Z_o(z) = 1_m + \sum_{k=1}^{n} 2C(k) z^k + 0(z^{n+1}) \quad . \tag{116}$$

Furthermore, N is unique up to a frequency-insensitive 2m-port all-pass cascaded on its output-side.

Proof. Let us employ the procedure described in lemma 4 to construct $P_n(z)$ and $Q_n(z)$ from the prescribed data. Then, as we know,

$$\det P_n(z) \cdot \det Q_n(z) \neq 0 \ , \quad |z| \leq 1 \quad , \tag{117}$$

and $Z_o(z) \equiv 2P_n^{-1}(z) Q_n(z)$ is positive.

As before, let

$$G_1(z) = \frac{1}{2} P_n(z) + Q_n(z) \tag{118}$$

and

$$H(z) = \frac{1}{2} P_n(z) - Q_n(z) \quad . \tag{119}$$

Since property I_2 is satisfied,

$$G_1(z)G_{1*}(z) = 1_m + H(z)H_*(z) \tag{120}$$

and it remains to prove that $G_1(z)$ is minimum-phase.

Obviously, the positivity of $Z_o(z)$ implies that of $1_m + Z_o(z)$ and this entails that the latter be nonsingular in $|z| < 1$. But in view of (118),

$$\det(1_m + Z_o(z)) = 2^m \cdot \frac{\det G_1(z)}{\det P_n(z)} \tag{121}$$

and it follows that $\det G_1(z)$ is free of zeros in $|z| < 1$.[11]

The construction of N is now completed by determining $G_2(z)$ as the minimum-phase solution of (18). Of course, the arbitrary constant unitary matrix right-multiplier V can again be absorbed in an output 2m-port all-pass, Q.E.D.

Comment 3. The recursion in (98) was discovered (in the scalar case) by the author and published for the first time in Reference 7. It is another manifestation of the incredibly rich substructure exhibited by block-Toeplitz matrices.

It is also important to understand that the integer n is det[illegible]mined by the number of data and serves only as an upper bound on the degrees of $P_n(z)$ and $Q_n(z)$. For example, the set $C(k) = 0_m$, $k = 1 \to n$, admits the SIP since

$$T_n = 1 > 0 \quad . \tag{122}$$

[11] Actually, $\det G_1(z) \neq 0$ in $|z| \leq 1$.

Clearly, in this case

$$(T_n^{-1})_{1;} = [1_m \; 0_m \; . \; . \; . \; 0_m] \tag{123}$$

and

$$E_n = [0_m \; 0_m \; . \; . \; . \; 0_m] \tag{124}$$

so that

$$A_0 = 1_m \tag{125}$$

and

$$P_n(z) = 1_m = 2Q_n(z) \tag{126}$$

are compatible choices. Thus, $P_n(z)$ and $Q_n(z)$ are constants and total degree reduction has occurred.

At this stage we are ready to state and prove our first major theorem which serves as the foundation stone for the multichannel spectral estimation technique developed in the next section. However, we must first insert another definition.

Def. 2. A matrix Z(z) is said to be full-passive if it is positive and

$$K(\theta) \equiv \frac{Z(e^{j\theta}) + Z^*(e^{j\theta})}{2} \tag{127}$$

is nonsingular for almost all θ in $-\pi \leq \theta \leq \pi$; i.e., if

$$\det K(\theta) \neq 0 \;, \quad \text{a.e.} \tag{128}$$

Similarly, an mxm matrix $\Gamma(z)$ is full-bounded if it is bounded and

$$\det(1_m - \Gamma^*(e^{j\theta})\Gamma(e^{j\theta})) \neq 0 \;, \quad \text{a.e.} \tag{129}$$

Theorem 1. Let C(k), $k = 0 \to n$, be any given set of n+1 constant mxm matrices such that $C(0) = 1_m$ and $T_n > 0$, and let N denote the (unique) associated interpolatory 2m-port defined in the corollary to lemma 4.

Let $\mathcal{P}_n$ stand for the collection of all mxm full-passive matrices

$$Z(z) = 1_m + \sum_{k=1}^{n} 2C(k)z^k + 0(z^{n+1}) \quad . \tag{130}$$

(Any such Z(z) is said to interpolate to the prescribed data.)

Then, the class $\mathcal{P}_n$ coincides with the set of all mxm positive matrices Z(z) generated by formulas (45) and (37) in which $\rho(z)$ is an arbitrary mxm full-bounded matrix.

Proof. For n=0, $\mathcal{P}_0$ is the set of all full-passive mxm matrices Z(z) such that $Z(0) = 1_m$. Hence, in particular,

$$S(z) = (Z(z) - 1_m)(Z(z) + 1_m)^{-1} \tag{131}$$

is bounded and equal to 0_m for $z = 0$. Thus, by Schwartz's lemma,[12]

$$S(z) = z\rho(z) \quad , \tag{132}$$

$\rho(z)$ bounded.

In other words, all positive Z's that satisfy the interpolation constraint $Z(0) = 1_m$ are obtained by letting $\rho(z)$ range over all bounded mxm matrices in the formula,

$$Z = (1_m + z\rho)(1_m - z\rho)^{-1} \quad . \tag{133}$$

But Eq. (132) is (37) with $H(z) = 0_m$ and $G_1(z) = G_2(z) = 1_m$ so that the corresponding interpolatory 2m-port N has the normalized scattering description,

$$S_n(p) = \left[\begin{array}{c|c} 0_m & 1_m \\ \hline 1_m & 0_m \end{array}\right] \quad . \tag{134}$$

Algebraic manipulation of Eq. (133) yields,

$$(1_m - e^{j\theta}\rho(e^{j\theta}))^* K(\theta)(1_m - e^{j\theta}\rho(e^{j\theta})) = 1_m - \rho^*(e^{j\theta})\rho(e^{j\theta}) \quad . \tag{135}$$

---

[12]Let $\Gamma(z)$ be an mxm bounded matrix such that $\Gamma(0) = 0_m$. Then [9], $\Gamma(z)/z$ is also bounded.

Consequently, $Z(z)$ is full-passive iff $\rho(z)$ is full-bounded and the case $n=0$ is verified. For $n \geq 1$ we proceed by induction.

First, $Z(z) \in \mathcal{P}_n$ obviously implies that $Z(z) \in \mathcal{P}_{n-1}$.[12'] Let the 2m-port interpolatory cascade N_1 constructed with the data $C(k)$, $k=0 \rightarrow n-1$, be parametrized as described in lemma 1 by the mxm polynomial matrix $H_1(z)$.[13] Of course, degree $H_1(z) \leq n-1$ and $H_1(0) = 0_m$. Let $G_{11}(z)$ and $G_{12}(z)$ denote the companion left-right mxm polynomial Wiener-Hopf factors:

$$G_{11}G_{11*} = 1_m + H_1H_{1*} \quad , \tag{136}$$

$$G_{12}G_{12*} = 1_m + H_{1*}H_1 \quad . \tag{137}$$

Also, let

$$L_1(z) = G_{11}^{-1}(z)H_1(z)G_{12}(z) \quad . \tag{138}$$

The induction hypothesis permits us to assert that

$$Z = (1_m + S)(1_m - S)^{-1} \tag{139}$$

where (Eq. (37)),

$$S = (L_1 + z\tilde{G}_{11}\eta)(G_{12} + z\tilde{H}_1\eta)^{-1} \tag{140}$$

and $\eta(z)$ is some full-bounded mxm matrix. Thus, making the appropriate substitutions in Eq. (44) and replacing n by n-1 we obtain,

$$\frac{S(z) - S_{1o}(z)}{z^n} = G_{11}^{-1}\eta(G_{12} + z\tilde{H}_1\eta)^{-1} \tag{141}$$

in which

$$S_{1o}(z) = G_{11}^{-1}(z)H_1(z) \quad . \tag{142}$$

According to lemma 2, the power-series expansions of $S(z)$ and $S_{1o}(z)$ about $z=0$ agree up to z^{n-1} and this means that the quotient on the left-hand

---

[12'] $T_n > 0$ implies that $T_{n-1} > 0$. Incidentally, we already know from lemma 4 that $\mathcal{P}_n$ is not empty.

[13] This $H_1(z)$ must not be confused with the constant matrix H_1 appearing in (8).

side of (141) is analytic in $|z| < 1$. However, (139) determines the power-series expansion of $S(z)$ up to z^n because that of $Z(z)$ is known up to z^n.

If we couple this with the observation that $S_{1o}(z)$ is completely fixed by the data $C(k)$, $k = 0 \to n-1$, it is concluded from (141) that

$$\eta(0) = G_{11}(0) \cdot \left.\frac{S(z) - S_{1o}(z)}{z^n}\right|_{z=0} \cdot G_{12}(0) \tag{143}$$

is a <u>known</u> quantity. Hence, $\eta(z)$ must be selected from the class of mxm bounded matrices which at $z = 0$ interpolate to the <u>prescribed</u> value given in (143).

Clearly, $\eta(z)$ bounded implies that $\|\eta(0)\| \leq 1$ but actually, $\|\eta(0)\| < 1$. Suppose otherwise. Then, $1_m - \eta^*(0)\eta(0)$ is singular and there exists a vector $\underset{\sim}{a}$ of unit norm such that

$$\underset{\sim}{a} = \eta^*(0)\eta(0)\underset{\sim}{a} \quad . \tag{144}$$

Or,

$$\underset{\sim}{a} = \eta^*(0)\underset{\sim}{b} \tag{145}$$

where

$$\underset{\sim}{b} = \eta(0)\underset{\sim}{a} \quad . \tag{146}$$

Evidently,

$$\underset{\sim}{a}^*\underset{\sim}{a} = \underset{\sim}{b}^*\underset{\sim}{b} = 1 \quad . \tag{147}$$

Construct two constant mxm unitary matrices V and U which incorporate $\underset{\sim}{a}$ and $\underset{\sim}{b}$, respectively, into their first columns and let

$$\delta(z) = U^*\eta(z)V \quad . \tag{148}$$

The matrix $\delta(z)$ is mxm and bounded and $\delta_{1,1}(0) = 1$.

Since all elements of $\delta(z)$ are analytic and bounded by unity in $|z| < 1$, we conclude by maximum-modulus that $\delta_{1,1}(z) \equiv 1$. In turn, the requirement

$$1_m - \delta^*(z)\delta(z) \geq 0 \quad , \quad |z| < 1 \quad , \tag{149}$$

then forces all the other entries in the first row and column of $\delta(z)$ to be identically zero so that

$$\delta(z) = 1_1 \dotplus \delta_1(z) \quad , \tag{150}$$

$\delta_1(z)$ an $(m-1)x(m-1)$ bounded matrix. Thus,

$$1_m - \eta^* \eta = V^*(1_m - \delta^* \delta)V = V^*(0_1 \dotplus (1_{m-1} - \delta_1^* \delta_1))V \tag{151}$$

is clearly singular for all $|z| \leq 1$ which contradicts the full-bounded character of $\eta(z)$. Consequently,

$$1_m - \eta^*(0)\eta(0) > 0 \tag{152}$$

and the renormalization scheme described in Appendix B becomes available.

Let R_a and R_b be any pair of hermitean matrix solutions of the Eqs.,

$$R_a^2 = 1_m - \eta(0)\eta^*(0) \quad , \tag{153}$$

$$R_b^2 = 1_m - \eta^*(0)\eta(0) \quad . \tag{154}$$

Then, from Schwartz's lemma and the renormalization theorem in Appendix B,

$$R_a(1_m - \eta\eta^*(0))^{-1}(\eta - \eta(0))R_b^{-1} = z\rho(z) \quad , \tag{155}$$

$\rho(z)$ an mxm bounded matrix. In addition, it is seen from Eq. (B10) that $\eta(z)$ is full-bounded iff $\rho(z)$ if full-bounded. The last step is to solve (155) for $\eta(z)$, substitute into (140) and then rearrange properly.

The end result is

$$S = (L + z\tilde{G}_1\rho)(G_2 + z\tilde{H}\rho)^{-1} \tag{156}$$

where

$$H = R_a^{-1}(H_1 + z\eta(0)\tilde{G}_{12}) \; , G_1 = R_a^{-1}(G_{11} + z\eta(0)\tilde{L}_1) \; , \tag{157}$$

$$L = (L_1 + z\tilde{G}_{11}\eta(0))R_b^{-1} \; , \; G_2 = (G_{12} + z\tilde{H}_1\eta(0))R_b^{-1} \quad . \qquad (158)$$

The proof that $G_1(z)$ and $G_2(z)$ are minimum-phase is easy. For example, since

$$G_1 = R_a^{-1}(1_m + z\eta(0)\tilde{L}_1 G_{11}^{-1})G_{11} \quad , \qquad (159)$$

it is clear that G_1 is minimum-phase iff the middle factor in (159) is minimum-phase. But $L_1 = G_{11*}H_1G_{12*}^{-1}$ so that $\tilde{L}_1 = G_{12}^{-1}\tilde{H}_1G_{11}$ and

$$\tilde{L}_1 G_{11}^{-1} = G_{12}^{-1}\tilde{H}_1 \quad . \qquad (160)$$

However, from (137) and maximum-modulus it is seen that $G_{12}^{-1}(z)\tilde{H}_1(z)$ is bounded. Thus, since $\|z\eta(0)\| < 1$, for $|z| \le 1$,

$$1_m + z\eta(0)\tilde{L}_1(z)G_{11}^{-1}(z) \qquad (161)$$

is a positive matrix and its determinant is necessarily devoid of zeros in $|z| < 1$. A similar proof applies of course to $G_2(z)$.

Equations (157) and (158) can be combined into the single matrix version,

$$\left[\begin{array}{c|c} G_1 & H \\ \hline L_* & G_{2*} \end{array}\right] = T(z) \cdot \left[\begin{array}{c|c} G_{11} & H_1 \\ \hline L_{1*} & G_{12*} \end{array}\right] \qquad (162)$$

where

$$T(z) = (R_a^{-1} \dotplus R_b^{-1}) \cdot \left[\begin{array}{c|c} 1_m & z^n\eta(0) \\ \hline z^{-n}\eta^*(0) & 1_m \end{array}\right] \quad . \qquad (163)$$

Let

$$\sigma_P \equiv \left[\begin{array}{c|c} 1_m & 0_m \\ \hline 0_m & -1_m \end{array}\right] \quad .$$

Then, by direct verification,

$$T \sigma_P T_* = \sigma_P \tag{164}$$

and

$$L_{1*}L_1 - G_{12*}G_{12} = G_{12*}H_{1*}G_{11*}^{-1}G_{11*}H_1G_{12*}^{-1} - G_{12*}G_{12}$$

$$= G_{12*}(H_{1*}H_1 - G_{12}G_{12*})G_{12*}^{-1} = -1_m \quad . \tag{164a}$$

Hence,

$$\left[\begin{array}{c|c} G_1 & H \\ \hline L_* & G_{2*} \end{array}\right] \sigma_P \left[\begin{array}{c|c} G_1 & H \\ \hline L_* & G_{2*} \end{array}\right]_* = \sigma_P \quad ; \tag{165}$$

i.e.,

$$G_1G_{1*} = 1_m + HH_* \quad , \tag{166}$$

$$G_2G_{2*} = 1_m + H_*H \tag{167}$$

and

$$L = G_1^{-1}HG_2 = G_{1*}HG_{2*}^{-1} \quad . \tag{168}$$

In short, it has been shown that every member of $\mathcal{P}_n$ can be synthesized by an appropriate full-bounded termination of the output-side of the 2m-port N defined by H(z).[14] Since part 3) of lemma 3 assures us that this 2m-port is an essentially unique construct from the full set of coefficients C(0), C(1), ..., C(n), the proof of theorem 1 is complete, Q.E.D.

In the next section the procedure developed in theorem 1 will be given a condensed algorithmic formulation and then applied to the problem of spectral estimation.

[14] Note that degree $H(z) \leq n$ and $H(0) = 0_m$.

IV. THE CLASS OF ALL SPECTRAL ESTIMATORS

Let us now suppose that we have available n+1 error-free mxm covariance samples C(0), C(1), ..., C(n) of an m-dimensional full-rank random vector process $\underset{\sim}{x}_t$. We shall produce a parametric formula for the class of all spectral densities $K(\theta)$ consistent with the given data.

Theorem 2 (The Master Result). Let $\underset{\sim}{x}_t$ denote a full-rank m-dimensional random vector-process and suppose that its first n+1 matrix mxm covariance samples C(0), C(1), ..., C(n) are known without error.

Let T_n and $T_n(-1)$ denote the Toeplitz matrices constructed on C(0), C(1), ..., C(n) and on the "reversed" data C(0), C(-1), ..., C(-n), respectively. (Clearly, $T_n(-1)$ is obtained by replacing each block C(k) in T_n by C(-k), $k=0 \to n$.) Let

$$\underset{\sim}{\xi}(z) = (1_m \; z1_m \; . \; . \; . \; z^n 1_m)' \tag{169}$$

Then, the following 3-step algorithm generates all spectral density matrices $K(\theta)$ compatible with the given data.

1) Affect the (unique) Gauss factorizations,

$$T_n = \Delta_n^* \Delta_n \; , \quad T_n(-1) = \Delta_n^*(-1)\Delta_n(-1) \tag{170}$$

where Δ_n and $\Delta_n(-1)$ are both lower-triangular with positive diagonal scalar entries. Let M_n and $M_n(-1)$ represent the two submatrices formed with the first m columns of Δ_n^{-1} and $\Delta_n^{-1}(-1)$, respectively.

2) Define two mxm polynomial matrices

$$P_n(z;1) = M_n^* \underset{\sim}{\xi}(z) \tag{171}$$

and

$$P_n(z;-1) = \underset{\sim}{\xi}'(z)M_n(-1) \; , \tag{172}$$

of degrees $\leq$ n and let[15]

[15] $\tilde{P}_n(z;-1) = z^n P_{n*}(z;-1)$.

$$D_n(z) \equiv P_n(z;1) - z\rho(z)\tilde{P}_n(z;-1) \quad . \tag{173}$$

3) Then,[16]

$$K(\theta) = D_n^{-1}(e^{j\theta})(1_m - \rho(e^{j\theta})\rho^*(e^{j\theta}))D_{n*}^{-1}(e^{j\theta}) \tag{174}$$

where $\rho(z)$ is an arbitrary full-bounded mxm function.

Proof. Assume that $C(0) = 1_m$ and let us show that $P_n(z;1) = P_n(z)$. As we known, $P_n(z) = X\xi(z)$ where

$$X = (A_0^*)^{-1} \cdot (T_n^{-1})_{1;} \tag{175}$$

and

$$A_0^* A_0 = (1_m - E_n T_{n-1}^{-1} E_n^*)^{-1} \quad . \tag{176}$$

But clearly, from $T_n^{-1} = \Delta_n^{-1}(\Delta_n^*)^{-1}$ and the lower-triangular character of Δ_n, it follows immediately that

$$(T_n^{-1})_{1;} = \Delta_{00}^{-1} \cdot M_n^* \tag{177}$$

where Δ_{00} is the upper left-hand corner mxm piece in Δ_n. Furthermore, since the matrix on the right-hand side of (176) is the upper left-hand corner mxm piece in T_n^{-1}, we can also make the identification,

$$A_0^* = \Delta_{00}^{-1} \quad . \tag{178}$$

Thus,

$$X = \Delta_{00} \cdot \Delta_{00}^{-1} \cdot M_n^* = M_n^* \tag{179}$$

and $P_n(z) = M_n^* \xi(z) = P_n(z;1)$. To arrive at an understanding of the meaning of $P_n(z;-1)$ we must derive a general expression for $K(\theta)$.

Let $C(k)$ denote the covariance function of the full-rank process x_t. Then necessarily,

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[16] $D_{n*}(e^{j\theta}) = D_n^*(e^{j\theta})$, θ real.

$$T_k > 0 \quad \text{and} \quad T_k(-1) > 0 \quad , \tag{180}$$

$$k = 0 \to \infty \quad .$$

Hence, invoking the multivariable Schur theorem,

$$Z(z) = 1_m + \sum_{k=1}^{\infty} 2C(k)z^k \tag{181}$$

is a positive mxm matrix.

Clearly,

$$K(\theta) \sim \sum_{k=-\infty}^{\infty} C(k)e^{jk\theta} = \frac{Z(e^{j\theta}) + Z^*(e^{j\theta})}{2} \tag{182}$$

where

$$Z(e^{j\theta}) \equiv \underset{r\to 1-0}{\text{limit}}\, Z(re^{j\theta}) \quad . \tag{183}$$

To describe all spectral densities $K(\theta)$ consistent with the data it is sufficient to exhibit a formula for the hermitean parts of all full-passive mxm matrices,

$$Z(z) = 1_m + \sum_{k=1}^{\infty} 2C(k)z^k + 0(z^{n+1}) \quad . \tag{184}$$

As a preliminary step we shall use lemma 2 to generate all corresponding mxm bounded functions,

$$S(z) = (Z(z) - 1_m)(Z(z) + 1_m)^{-1} \quad . \tag{185}$$

Indeed, every such S(z) is given by Eq. (37) with $\rho(z)$ an arbitrary full-bounded mxm matrix. Nevertheless, it turns out to be more convenient to work with the equally valid expression

$$S(z) = (G_1 + z\rho\tilde{L})^{-1}(H + z\rho\tilde{G}_2) \quad , \tag{186}$$

that is obtained by reducing

$$S = S_{11} + S_{12}(1_m - \Gamma S_{22})^{-1}\Gamma S_{21} \qquad (187)$$

instead of (26).

A straightforward calculation with the aid of (186) yields, for $|z| = 1$,

$$(G_1 + z\rho\tilde{L})(1_m - SS^*)(G_1 + z\rho\tilde{L})^* = 1_m - \rho\rho^* \ . \qquad (188)$$

Or, since

$$\frac{Z + Z^*}{2} = (1_m - S)^{-1}(1_m - SS^*)(1_m - S^*)^{-1} \qquad (189)$$

and

$$1_m - S = (G_1 + z\rho\tilde{L})^{-1}(G_1 - H - z\rho(\tilde{G}_2 - \tilde{L})) \ , \qquad (190)$$

we obtain

$$\frac{Z + Z^*}{2} = D_n^{-1}(1_m - \rho\rho^*)D_{n*}^{-1} \ , \quad |z| = 1 \ , \qquad (191)$$

where

$$D_n(z) = G_1(z) - H(z) - z\rho(z)(\tilde{G}_2(z) - \tilde{L}(z)) \ . \qquad (192)$$

Hence,

$$D_n(z) = P_n(z;1) - z\rho(z)\tilde{P}_n(z;-1) \qquad (193)$$

if it can be demonstrated that

$$P_n(z;-1) \equiv G_2(z) - L(z) \qquad (194)$$

agrees with (172).

From Eqs. (25) and (39),

$$LG_2^{-1} = G_1^{-1}H = S_o \qquad (195)$$

so that

$$Z_o = (1_m + S_o)(1_m - S_o)^{-1} = (G_2 + L)(G_2 - L)^{-1} \quad . \tag{196}$$

Thus,

$$\frac{Z_o + Z_{o*}}{2} = ((G_2 - L)(G_{2*} - L_*))^{-1} \quad ; \tag{197}$$

i.e.,

$$(\tilde{G}_2 - \tilde{L}) \cdot \frac{Z_o + Z_{o*}}{2} = z^n \cdot (G_2 - L)^{-1} \quad . \tag{198}$$

Since

$$G_{2*}(z)G_2(z) = 1_m + L_*(z)L(z) \quad , \tag{199}$$

it follows that $\|S_o(z)\| < 1$, $|z| \leq 1$. Hence,

$$(1_m - S_o(z))G_2(z) = G_2(z) - L(z) \tag{200}$$

is nonsingular in $|z| \leq 1$ and the exact same reasoning used in lemma 3 to derive (171) also yields (172) when applied to (198).

To complete the proof we must remove the restriction $C(0) = 1_m$ and this is accomplished by exploiting the following series of observations.

O_1. There exists a lower-triangular matrix R_o with positive diagonal entries such that

$$(R_o^*)^{-1}C(0)R_o^{-1} = 1_m \quad . \tag{201}$$

O_2. The use of (174) in conjunction with the normalized data

$$(R_o^*)^{-1}C(k)R_o^{-1} \; , \quad k = 0 \to n \quad , \tag{202}$$

is tantamount to reconstructing all admissible normalized spectral densities,

$$(R_o^*)^{-1}K(\theta)R_o^{-1} \quad . \tag{203}$$

O_3. The lower-triangular factorizations associated with the "forward" and "reverse-direction" Toeplitz matrices constructed on these normalized samples equal, respectively, $\Delta_n(1)$ and $\Delta_n(-1)$ multiplied on the right by

$$\Lambda_o^{-1} = (R_o \dot{+} R_o \dot{+} \ldots \dot{+} R_o)^{-1} \quad . \tag{204}$$

Thus,

$$M_n(1) \to \Lambda_o M_n(1) \quad , \tag{205}$$

$$M_n(-1) \to \Lambda_o M_n(-1) \quad , \tag{206}$$

$$P_n(z;1) \to M_n^*(1)\Lambda_o^* \xi(z) \quad , \tag{207}$$

$$P_n(z;-1) \to \xi'(z)\Lambda_o M_n(-1) \tag{208}$$

and

$$D_n(z) \to M_n^*(1)\Lambda_o^* \xi(z) - z\rho(z) M_n^*(-1)\Lambda_o^* \tilde{\xi}'(z) \equiv \hat{D}_n(z) \quad . \tag{209}$$

O_4. Since

$$(R_o^*)^{-1} K R_o^{-1} = \hat{D}_n^{-1}(1_m - \rho\rho^*)\hat{D}_{n*}^{-1} \quad , \tag{210}$$

we see that

$$K = \underline{D}_n^{-1}(1_m - \rho\rho^*)\underline{D}_{n*}^{-1} \tag{211}$$

where

$$\underline{D}_n(z) = \hat{D}_n \cdot (R_o^*)^{-1} = M_n^*(1)\xi(z) - z\rho(z)M_n^*(-1)\tilde{\xi}'(z) = P_n(z;1) - z\rho(z)\tilde{P}_n(z;-1) = D_n(z) \quad , \tag{212}$$

Q.E.D.

Comment 4. The simplicity of the general formula (174) is truly remarkable, especially when it is realized that $P_n(z;1)$ and $P_n(z;-1)$ are intimately related to the right and left matrix orthogonal polynomials defined by the sequence $C(k)$, $k = 0 \to n$. This means [6, 10] that both can be generated recursively (and simultaneously) by readily available and highly efficient algorithms of the Levinson type, thereby obviating any need to compute the inverses Δ_n^{-1} and $\Delta_n^{-1}(-1)$.

Although we intend to resume our studies of these matters in depth we shall nevertheless give the explicit connection formulas for the benefit of the reader who wishes to proceed directly to the numerical implementation of (174).

Namely, if $\Delta_{00}^{(n)}$ and $\Delta_{00}^{(n)}(-1)$ denote the upper left-hand corner mxm blocks in Δ_n and $\Delta_n(-1)$, respectively, and if $Q^{(n)}(z)$ and $R^{(n)}(z)$ represent, in the same order, the monic left and right degree-n matrix orthogonal polynomials alluded to above, then [6],

$$Q^{(n)}(z) = \tilde{P}_n(z;1)\Delta_{00}^{(n)} \tag{212a}$$

and

$$R^{(n)}(z) = \Delta_{00}^{(n)*}(-1)\tilde{P}_n(z;-1) \quad . \tag{212b}$$

Consequently, Eqs. (136)-(142) in Reference 6 become immediately available after a suitable 1-1 map of some of the symbols.

V. THE MULTICHANNEL FLAT-ECHO ESTIMATOR

Under the assumption that $\underset{\sim}{x}_t$ is a full-rank random vector-process with absolutely continuous spectral function, a knowledge of its covariance samples C(0), C(1), ..., C(n) permits us to assert that all admissible interpolatory $K(\theta)$ are given by (174) where $\rho(z)$ is any full-bounded mxm function.

Evidently, as is seen from this expression, $K(\theta)$ satisfies Paley-Wiener iff the same is true of

$$1_m - \rho(e^{j\theta})\rho^*(e^{j\theta}) \quad .$$

This granted, there exists [6] an mxm bounded matrix function $B_\rho(z)$ which is analytic together with its inverse in $|z| < 1$, such that

$$1_m - \rho(e^{j\theta})\rho^*(e^{j\theta}) = B_\rho(e^{j\theta})B_\rho^*(e^{j\theta}) \tag{213}$$

for almost all θ in $-\pi \leq \theta \leq \pi$.

From (213) and (174) it follows that

$$K(\theta) = B(e^{j\theta})B^*(e^{j\theta}) \tag{214}$$

where

$$B(z) = D_n^{-1}(z)B_\rho(z) \tag{215}$$

is also analytic together with its inverse in $|z| < 1$. Hence, formula (215) constitutes a complete parametrization of the set of all admissible spectral density Wiener-Hopf factors consistent with the given data, in terms of the free bounded matrix-function parameter $B_\rho(z)$.

Since[17]

$$B(0) = D_n^{-1}(0)B_\rho(0) = P_n(0;1)B_\rho(0) = P_n(0)B_\rho(0) \tag{216}$$

and

$$1_m - B_\rho(0)B_\rho^*(0) \geq 0 \quad , \tag{217}$$

[17] Remember that $P_n(z)$ and $P_n(z;1)$ are identical.

we see that

$$P_n(0)P_n^*(0) - B(0)B^*(0) = P_n(0)(1_m - B_\rho(0)B_\rho^*(0))P_n^*(0) \geq 0 \ . \quad (218)$$

In particular,[18]

$$|\det P_n(0)| \geq |\det B(0)| \ . \quad (219)$$

Furthermore, because $P_n(0)$ is nonsingular, equality in (219) is possible iff $B_\rho(0)$ is a unitary matrix. But then, it follows routinely from maximum-modulus that $B_\rho(z) = B_\rho(0)$ for all z and this in turn implies that

$$\rho(z) \equiv 0_m \ . \quad (220)$$

According to (173) and (174), $\rho(z) \equiv 0_m$ corresponds to the choice of estimator

$$K_{ME}(\theta) = (P_n^*(e^{j\theta})P_n(e^{j\theta}))^{-1} \quad (221)$$

and invoking the result[19]

$$2\ln|\det P_n(0)| + 2\ln|\det B_\rho(0)| = 2\ln|\det B(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \det K(\theta)d\theta \equiv \text{Entropy}(K) \ , \quad (222)$$

it is immediately concluded that $K_{ME}(\theta)$ is the maximum-entropy estimator [11]. In other words, $K_{ME}(\theta)$ is obtained by the termination of the basic interpolatory 2m-port N in a matched m-port load, $\Gamma(z) \equiv 0_m$.

Alternatively, it is seen from (174) that the entropy of any admissible $K(\theta)$ associated with the choice of bounded function $\rho(z)$ is given by,[20]

$$\text{Entropy}(K) = \text{Entropy}(K_{ME}) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \det^{-1}(1_m - \rho(e^{j\theta})\rho^*(e^{j\theta}))d\theta \leq \text{Entropy}(K_{ME}) . \quad (223)$$

[18] If A and B are both hermitean nonnegative-definite, $\det(A+B) \geq \det A + \det B$.

[19] Apply Cauchy's theorem to the analytic function $\ln \det B(z)$ and observe that

$$\det K(\theta) = |\det B(e^{j\theta})|^2 \ .$$

[20] Since $1_m - \rho(e^{j\theta})\rho^*(e^{j\theta}) \geq 0$, a.e., the reciprocal of its determinant is a number which is greater than unity for almost all θ.

Clearly, equality is possible iff $\rho(e^{j\theta}) = 0_m$ a.e. in $-\pi \le \theta \le \pi$; i.e., iff $K(\theta) = K_{ME}(\theta)$.

In view of (223), it is legitimate to define

$$\frac{1}{2\pi} \ln \det^{-1}(1_m - \rho(e^{j\theta})\rho^*(e^{j\theta})) \tag{224}$$

to be the entropy <u>echo-loss</u> density at angle θ. A <u>flat-echo</u> estimator (an FEE) is one for which this density is a constant <u>independent</u> of θ.

One particularly simple (and numerically effective way) to generate FEE's is to choose

$$\rho(z) = \mu d(z) \tag{225}$$

where μ is an arbitrary <u>constant</u> mxm matrix such that

$$1_m - \mu\mu^* > 0 \quad , \tag{226}$$

and d(z) is an arbitrary <u>rational</u> mxm regular paraconjugate unitary matrix.[21]

Thus, for this <u>subset</u> of FEE's,

$$K_{FE}(\theta) = D_n^{-1}(e^{j\theta})(1_m - \mu\mu^*)D_{n*}^{-1}(e^{j\theta}) \tag{227}$$

where

$$D_n(z) = P_n(z) - z\mu d(z)\tilde{P}_n(z;-1) \quad , \tag{228}$$

and

$$\text{Entropy}(K_{FE}) = \text{Entropy}(K_{ME}) - \ln \det^{-1}(1_m - \mu\mu^*) \quad . \tag{229}$$

Note, that the entropy of $K_{FE}(\theta)$ can be <u>fixed in advance</u> by an appropriate assignment of μ and d(z) can then be designed to accomplish the necessary tuning. The latter emerges as a problem of interpolation with regular mxm matrix all-passes.

Although a complete (recursive) solution has already been obtained for the case m = 1 [1], much work remains to be done for general m.

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[21] $d(z)d_*(z) = d_*(z)d(z) = 1_m$ and d(z) is analytic in $|z| \le 1$.

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APPENDIX A

The Multivariable Schur Theorem. The power series

$$Z(z) = 1_m + \sum_{k=1}^{\infty} 2C(k)z^k \tag{A1}$$

defines an mxm positive matrix in $|z| < 1$, iff

$$T_k \geq 0 \ , \quad k = 1 \to \infty \ . \tag{A2}$$

Proof. Suppose that Z(z) is positive. Then, by definition, Z(z) is analytic in $|z| < 1$ and

$$\frac{Z(z) + Z^*(z)}{2} \geq 0 \ , \quad |z| < 1 \ . \tag{A3}$$

Hence, for every fixed r, $0 < r < 1$, and for every mxm trigonometric polynomial matrix

$$f(\theta) = f_0 + f_1 e^{-j\theta} + \dots + f_k e^{-jk\theta} \ , \tag{A4}$$

θ real, $k \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(\theta) \cdot \frac{Z(re^{j\theta}) + Z^*(re^{j\theta})}{2} \cdot f(\theta) d\theta \geq 0 \ . \tag{A5}$$

Or, carrying out the integration,

$$\sum_{\ell, q=0}^{k} r^{|\ell - q|} \cdot f_\ell^* C(q-\ell) f_q \geq 0 \tag{A6}$$

where $C(0) \equiv 1_m$.

In the limit as $r \to 1-0$ with k held fixed, (A6) goes into

$$\sum_{\ell, q=0}^{k} f_\ell^* C(q-\ell) f_q \geq 0 \tag{A7}$$

and because of the arbitrary character of the mxm matrices $f_0, f_1, \cdots, f_k$ we conclude that $T_k \geq 0$.

Sufficiency. From $T_k \geq 0$ we deduce, in particular, that

$$\left[\begin{array}{c|c} 1_m & C(k) \\ \hline C^*(k) & 1_m \end{array}\right] \geq 0 \quad ; \tag{A8}$$

i.e., for every $k \geq 0$,

$$\| C(k) \| \leq 1 \quad . \tag{A9}$$

Thus, if $Z(z)$ is defined by (A1) we obtain the estimate,

$$\| Z(z) \| \leq 1 + \sum_{k=1}^{\infty} 2 \| C(k) \| \cdot |z|^k \leq \tag{A10}$$

$$\leq 1 + \sum_{k=1}^{\infty} 2 |z|^k = \frac{1 + |z|}{1 - |z|} < \infty \quad , \tag{A11}$$

$|z| < 1$. Consequently, $Z(z)$ is analytic in $|z| < 1$ and it only remains to demonstrate its positivity.

This can be achieved by observing [5] that

$$\frac{Z(z) + Z^*(z)}{2(1 - z\bar{z})} = \sum_{\ell=0}^{\infty} z^{\ell} \bar{z}^{\ell} \left(\sum_{k=0}^{\infty} C(k) z^k + \sum_{k=1}^{\infty} C(-k) \bar{z}^k \right) \tag{A12}$$

$$= \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} C(k) z^{\ell+k} \bar{z}^{\ell} + \sum_{\ell=0}^{\infty} \sum_{k=1}^{\infty} C(-k) z^{\ell} \bar{z}^{\ell+k} \tag{A13}$$

$$= \sum_{\beta=0}^{\infty} \sum_{\alpha=\beta}^{\infty} C(\alpha - \beta) z^{\alpha} \bar{z}^{\beta} + \sum_{\alpha=0}^{\infty} \sum_{\beta=\alpha+1}^{\infty} C(\alpha - \beta) z^{\alpha} \bar{z}^{\beta} \tag{A14}$$

$$= \sum_{\alpha,\,\beta=0}^{\infty} C(\alpha-\beta) z^{\alpha} \bar{z}^{\beta} = \lim_{k\to\infty} \sum_{\alpha,\,\beta=0}^{k} f_{\beta}^{*} C(\alpha-\beta) f_{\alpha} \geq 0 \tag{A15}$$

where

$$f_{\alpha} = z^{\alpha} 1_m \quad , \quad \alpha = 0 \to \infty \quad . \tag{A16}$$

(In reaching (A15) we have used (A7) and have also noted that the first sum in (A14) contains all the elements on and above the main diagonal of the doubly-infinite array

$$\{C(\alpha-\beta) z^{\alpha} \bar{z}^{\beta}\} \, , \quad \alpha, \beta = 0 \to \infty \quad , \tag{A17}$$

while the second contains all those strictly below.)

Since $1 - |z|^2 > 0$ for $|z| < 1$, (A15) yields

$$\frac{Z(z) + Z^{*}(z)}{2} \geq 0 \quad , \quad |z| < 1 \quad , \tag{A18}$$

and $Z(z)$ is positive, Q.E.D.

APPENDIX B

Our proof of theorem 1 made use of the following interesting result long known to network theorists.

The Renormalization Theorem. Let C_o be any constant mxm matrix such that $\|C_o\| < 1$. Let R_a and R_b denote, respectively, any pair of hermitean mxm matrix solutions of the two equations,

$$R_a^2 = 1_m - C_o C_o^* \tag{B1}$$

and

$$R_b^2 = 1_m - C_o^* C_o \quad . \tag{B2}$$

Then, if W(z) is an mxm bounded matrix, so is

$$\Gamma(z) = R_a(1_m - WC_o^*)^{-1}(W - C_o)R_b^{-1} \quad . \tag{B3}$$

Proof. The obvious identity

$$C_o^*(1_m - C_o C_o^*)^{-1} = (1_m - C_o^* C_o)^{-1} C_o^* \tag{B4}$$

yields,

$$C_o^* R_a^{-2} = R_b^{-2} C_o^* \quad . \tag{B5}$$

Hence, upon multiplying both sides of (B5) on the left by C_o we obtain

$$R_a^{-2} = 1_m + C_o R_b^{-2} C_o^* \tag{B6}$$

Similarly,

$$R_b^{-2} = 1_m + C_o^* R_a^{-2} C_o \quad . \tag{B7}$$

To establish the boundedness of $\Gamma(z)$ it is necessary to prove that

$$1_m - \Gamma(z)\Gamma^*(z) \geq 0 \quad , \quad |z| < 1 \quad . \tag{B8}$$

From (B3), (B1), and (B2) we obtain,[1]

$$(1_m - WC_o^*)R_a^{-1}(1_m - \Gamma\Gamma^*)R_a^{-1}(1_m - C_oW^*) = (1_m - WC_o^*)R_a^{-2}(1 - C_oW^*) - (W - C_o)R_b^{-2}(W^* - C_o^*)$$

$$= R_a^{-2} - WR_b^{-2}W^* - C_oR_b^{-2}C_o^* + WC_o^*R_a^{-2}C_oW^* +$$

$$+ C_oR_b^{-2}W^* - R_a^2C_oW^* + WR_b^{-2}C_o^* - WC_o^*R_a^{-2}$$

$$= 1_m - WW^* \quad . \tag{B9}$$

Thus,

$$1_m - \Gamma\Gamma^* = R_a(1_m - WC_o^*)^{-1}(1_m - WW^*)(1_m - C_oW^*)^{-1}R_a \geq 0 \tag{B10}$$

since $1_m - WW^* \geq 0$ (by hypothesis), Q.E.D.

The proof of the following impedance version is considerably simpler and is therefore omitted [8].

Corollary. *Let C_o be a constant mxm matrix such that*

$$R_o \equiv \frac{C_o + C_o^*}{2} > 0 \quad . \tag{B11}$$

Let $R_o^{1/2}$ equal the (unique) hermitean positive-definite square-root of R_o.

Then, if W(z) is an mxm positive matrix,

$$\Gamma(z) = R_o^{-1/2}(W - C_o)(W + C_o^*)^{-1}R_o^{1/2} \tag{B12}$$

is an mxm bounded matrix.

NOTE: By inverting (B12) we obtain

$$W(z) = R_o^{1/2}(1_m + \Gamma)(1_m - \Gamma)R_o^{1/2} + \frac{C_o - C_o^*}{2} \quad . \tag{B13}$$

In particular, if C_o is also hermitean, $R_o = C_o$ and

$$W(z) = C_o^{1/2}(1_m + \Gamma)(1_m - \Gamma)^{-1}C_o^{1/2} \quad . \tag{B14}$$

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[1] By starring both sides of (B5) we get $R_a^{-2}C_o = C_oR_b^{-2}$.

PART IV

Some Observations and Suggestions for New Work

by

Haywood E. Webb Jr.

IV. SOME OBSERVATIONS AND SUGGESTIONS FOR NEW WORK

The discussion on The Interpolatory Cascade, in Part I indicates that every admissible spectral estimator can be interpreted as the real part of the driving point impedance of a cascade of uniform transmission lines of characteristic impedances given by (63) terminated in an arbitrary load impedance W(Z) as shown in Fig 1. For given sample covariance data, it is a different load impedance that corresponds to a different spectral estimator. The FEE is a particular class of estimators which provides a set of "tuning" parameters. The setting of all the work here is in a discrete setting. It is the digital computer implementation that drives the discrete setting. We recall that a process with a continuous covariance function, in order to be non determinstic, must satisfy the continuous analogue of (7) Part 1.

$$\int \frac{|\operatorname{Log} A(w)|}{1+w^2} dw < \infty \tag{1}$$

Where A(w) is here the spectrum obtained from the Fourier transform of the covariance. But note also, that any function which satisfies (1) is not bandlimited, so that a discrete sampling representation introduces aliasing errors at the outset, and the problem, from a continuous viewpoint, is improperly formulated. The concept of entropy does not make sense.

In the network formulation of the spectral estimation problem, the discrete cascade, can be generalized to a distributed continuous transmission line. Again, we conjecture, that every admissible spectrum obtainable by a "continuation" of a segment of the covariance function, can be obtained by a non-uniform transmission line with a shunt capacitor, terminated in an arbitrary positive real load impedance. It would be interesting to maximize the continuous analogue (1) in this section to (7) in Part 1.

Note that under these circumstances that (1) of Part IV cannot become infinite under the assumptions. After all, (1) of Part IV is a necessary and sufficient condition that the inverse Fourier transform of A(w) not be quasi-analytic i.e., not be uniquely deterministic over its domain from its derivatives at a point.

It is proposed that further exploration of the continuous problem be made, and that after understanding it, that digital computer implementations be considered in the light of the results.

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